ADIC SPACES II

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Let's start by recalling some basic definitions.

- A Huber ring is a topological ring with an open subring of definition $A_0 \subset A$ adic with respect to a finitely generated ideal of definition.
- A is called *Tate* if it contains a topological nilpotent unit (pseudo-uniformizer) $g \in A$. A Huber ring A is *analytic* if $A^{\circ\circ}$ generates the unit ideal. Any Tate ring is analytic.
- A subring $A^+ \subset A$ is a *ring of integral elements* if it is open and integrally closed in A and $A^+ \subset A^\circ$. A Huber pair is a pair (A, A^+) of a Huber ring A and A^+ a ring of integral elements.
- The adic spectrum Spa(A, A^+) is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+| \leq 1$. For $x \in \text{Spa}(A, A^+)$, write $g \mapsto |g(x)|$ for a choice of corresponding valuation. The topology on Spa(A, A^+) is generated by open subsets of the form

  $$\{x : |f(x)| \leq |g(x)| \neq 0\}$$

  with $f, g \in A$. Rational subsets are

  $$U(T/s) := \{x \in X : |t(x)| \leq |s(x)| \neq 0, \text{for all } t \in T\}$$

  where $TA \subset A$ is open and $T \subset A$ is finite. It is a theorem of Huber for any rational subset $U \subset \text{Spa}(A, A^+)$, there is an universal complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ which moreover induces a homeomorphism onto U

  $$\text{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \text{Spa}(A, A^+).$$
- Define a pair of presheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$ of topological rings on Spa(A, A^+) by

  $$\mathcal{O}_X(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U), \qquad \mathcal{O}_X^+(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X^+(U).$$

  A Huber pair (A, A^+) is *sheafy* if $\mathcal{O}_X$ is a sheaf of topological rings.

1. Analytic adic spaces

We now say a few words about analytic adic spaces, which include all adic spaces that we shall see in Bhargav's class. These are spaces that are close to rigid analytic varieties but without finiteness assumptions.

Definition 1.1. Let (A, A^+) be a Huber pair. A point $x \in \text{Spa}(A, A^+)$ is *analytic* if the prime ideal supp$(x) := \{a \in A : |a(x)| = 0\}$ of A is *not* open. A point x of an adic space X is *analytic* if it has an affinoid neighborhood in which x is analytic. An adic space is *analytic* if all its points are analytic. In particular, the set of analytic points is open.

Proposition 1.2. *Let (A, A^+) be a complete Huber pair.*

(1) The Huber ring A is analytic if and only if all points of Spa(A, A^+) *are analytic.*
(2) A point $x \in$ Spa(A, A^+) *is analytic if and only if there is a rational neighborhood $U \subset$* Spa(A, A^+) *of x such that $\mathcal{O}_X(U)$ is Tate.*

Definition 1.3. A morphism $f : A \to B$ of Huber rings is adic if for one and hence any choice of rings of definition $A_0 \subset A$, $B_0 \subset B$ with $f(A_0) \subset B_0$, and $I \subset A_0$ an ideal of definition, $f(I)B_0$ is an ideal of definition. A morphism between Huber pairs is adic if $A \to B$ is.

Remark 1.4. If A is Tate, then any morphism $f : A \to B$ is.

Example 1.5. $\mathbf{Z}_p \to \mathbf{Z}_p[[x]]$ is not adic.

Proposition 1.6. *A map $(A, A^+) \to (B, B^+)$ of complete Huber pairs is adic if and only if* $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ *carries analytic points to analytic points.*

One may then define an adic morphism between adic spaces to be a morphism that sends analytic points to analytic points or equivalently is locally given by adic morphisms between Huber pairs.

Proposition 1.7. *If $(A, A^+) \to (B, B^+)$ is adic, then pullback along the associated map of topological spaces preserves rational subsets. Moreover, for a diagram $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ of Huber pairs where both morphisms are adic, let (A_0, B_0, C_0) be rings of definition compatible with the morphisms, and let $I \subset A_0$ be an ideal of definition. Let $D = B \otimes_A C$ and let D_0 be the image of $B_0 \otimes_{A_0} C_0$ in D. Make D into a Huber ring by declaring D_0 to be a ring of definition with ID_0 as its ideal of definition. Let D^+ be the integral closure of the image of $B^+ \otimes_{A^+} C^+$ in D. Then (D, D^+) is a Huber pair and it is the pushout of the diagram in the category of Huber pairs.*

Definition 1.8. A complete analytic Huber pair (A, A^+) is stably uniform if $\mathcal{O}_X(U)$ is uniform (A° is bounded) for all rational subsets $U \subset X = \mathrm{Spa}(A, A^+)$.

Theorem 1.9. *If the complete analytic Huber pair (A, A^+) is stably uniform, then it is sheafy. In particular, perfectoid spaces are sheafy.*

2. Rigid spaces

There is a fully faithful functor r from rigid analytic spaces over k to adic spaces over $\mathrm{Spa}(k, k^\circ)$ with the following properties:

- It sends $\mathrm{Sp}(A)$ to $\mathrm{Spa}(A, A^\circ)$ for affinoid A. And if $f : \mathrm{Sp}(A) \to \mathrm{Sp}(B)$ is a morphism induced by $\varphi : B \to A$, then $r(f)$ is given by the morphism $(B, B^\circ) \to (A, A^\circ)$ induced by φ. Here we might want to say a few words on how to topologize A. Take a presentation
$$A = k\langle T_i \rangle / I$$
with the residue seminorm
$$|\overline{f}| := \inf\{|h| : hI = \overline{f}\}.$$
This norm depends on the choice of presentation, but the topology it induces does not (by the open mapping theorem). This topologizes A and consequently
$$A^\circ = \mathcal{O}_k\langle T_i \rangle / (I \cap \mathcal{O}_k\langle T_i \rangle).$$
- An open embedding is sent to an open embedding.
- A family of admissible open subsets is an admissible covering if and only if their adic counterpart forms an honest open covering.
- The image of objects under r are locally of finite type over $\mathrm{Spa}(k, k^\circ)$.
- r induces an equivalence from the category of quasi-separated rigid analytic spaces over k to adic spaces over $\mathrm{Spa}(k, k^\circ)$ that are locally of finite type and quasi-separated.
- r commutes with fiber products and preserves étale and smooth morphisms.

3. Separated and proper morphisms

We already saw words like “locally of finite type,” “quasi-separated,” “étale" and “smooth." These are part of the foundational vocabulary of usual algebraic geometry. In this section we hope to convince you that there are reasonable analogues for adic spaces.

There is a usual notion of quasi-compactness of a spectral space. A spectral space is called quasi-separated if the intersection of any two quasi-compact open subsets is quasi-compact.

Definition 3.1. A ring morphism $f : A \to B$ from a Huber ring A to a complete Huber ring B is *of topologically finite type* if f is adic and there exists a finite subset $M \subset B$ such that $A[M]$ is dense in B, and there exist rings of definition A_0, B_0 of A and B and a finite subset $N \subset B_0$ such that $f(A_0) \subset B_0$ and $A_0[N]$ is dense in B_0.

Definition 3.2. Let $f : X \to Y$ be a morphism of adic spaces. Then f is called *locally of finite type* if for every $x \in X$ there exist open affinoid subspaces U, V of X, Y such that $x \in U$, $f(U) \subset V$ and the ring homomorphism of Huber pairs $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type. The morphism f is called *locally of weakly finite type* if one may choose U and V as above such that $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of topologically finite type. The morphism f is called of $^+$weakly finite type if f is quasi-compact and if there exists U and V as above together with a finite subset $E \subset \mathcal{O}_X(U)$ such that $\mathcal{O}_X^+(U)$ is the integral closure of $\mathcal{O}_Y^+[E \cup \mathcal{O}_X(U)^{\circ\circ}]$ in $\mathcal{O}_X(U)$. The morphism f is called *locally of finite presentation* if for every $x \in X$ there exist U, V as above such that $(\mathcal{O}_Y(V), \mathcal{O}_Y^+(V)) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is of topologically finite type and if the topology of $\mathcal{O}_Y(V)$ is discrete, the ring morphism $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ is of finite presentation.

Locally of weakly finite type morphisms are all adic morphisms. Morphisms between rigid analytic varieties are all of finite type.

Proposition 3.3. *Let $f : X \to Z$ and $g : Y \to Z$ be morphisms of adic spaces. Then the fiber product of f and g exists if either f is locally of finite type or if f is locally of weakly finite type and g is adic.*

Definition 3.4. A morphism of adic spaces $f : X \to Y$ is called *separated* if f is locally of weakly finite type and the image of the diagonal morphism $\Delta : X \to X \times_Y X$ is closed in $X \times_Y X$. The morphism f is called *universally closed* if f is locally of weakly finite type and for every adic morphism $Y' \to Y$ the projection $X \times_Y Y' \to Y'$ is closed. The morphism f is called *proper* if f is of $^+$weakly finite type, separated, and universally closed.

Recall that a point $y \in X$ of a topological space X called a generalization of point $x \in X$ (or x is called a specialization of y) if y is contained in every neighborhood of x in X.

In an analytic adic space X, all generalization of $x \in X$ form a chain. A point $x \in X$ is called maximal if it has no other generalizations. Then a point $x \in X$ is maximal if and only if the valuation v_x has rank 1. Let $x \in X$ be a generalization y, then the natural ring morphism $j : \mathcal{O}_{X,y} \to \mathcal{O}_{X,x}$ is local and flat and in particular injective. A morphism between analytic adic spaces $X \to Y$ sends maximal points to maximal points and f sends the set of all generalizations of $x \in X$ onto the set of all generalizations of $f(x) \in Y$.

Definition 3.5. Let $f : X \to Y$ be a morphism of adic spaces. Say that f is *specializing* at a point $x \in X$ if for every specialization y' of $f(x)$ in Y there exists a specialization x' of x in X with $y' = f(x')$. Say that f is *universally specializing* at a point $x \in X$ if f is locally of weakly finite type and for every adic morphism of adic spaces $Y' \to T$ and every point x' of $X \times_Y Y'$ lying over x, the projection $X \times_Y Y' \to Y'$ is specializing at x'. Say that f is specializing if f is universally specializing at every point of X and say that f is *universally specializing* if f is universally specializing at every point. Say that f is *partially proper* if f is locally of $^+$weakly finite type, spearated, and universally specializing.

Remark 3.6. A morphism f is proper if and only if it is partially proper and quasi-compact.

3.1. **The generic fiber construction.** Let's review how to extract the generic fiber of a p-adic formal scheme $\mathfrak{X}$. Let A be a complete $\mathbf{Z}_p$-algebra. Then A is a Huber ring with itself being a ring of definition and (p) being an ideal of definition and (A, A) is a Huber pair. Then the functor

$$\mathrm{Spf}(A) \mapsto \mathrm{Spa}(A, A)$$

extends to a fully faithful functor from p-adic formal schemes to adic spaces over $\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)$. This functor is denoted $\mathfrak{X} \mapsto \mathfrak{X}^{\mathrm{ad}}$. Let $\mathfrak{X}$ be a p-adic formal scheme, define its generic fiber to be

$$\mathfrak{X}_\eta := \mathfrak{X}^{\mathrm{ad}} \times_{\mathrm{Spa}(\mathbf{Z}_p, \mathbf{Z}_p)} \mathrm{Spa}(\mathbf{Q}_p, \mathbf{Z}_p).$$

By above, the fiber product always exists when $\mathfrak{X}$ is of topologically finite type over $\mathbf{Z}_p$. In the cases of our interests, the functor

$$\mathrm{Spf}\,(A) \mapsto \mathrm{Spa}(A[1/p], A)$$

always works as a fiber product.

4. The étale site

4.1. **Étale morphisms.** Let $f : X \to Y$ be a morphism between adic spaces that is locally of weakly finite type. Let $\Delta : X \to X \times_Y X =: Z$ be the diagonal morphism and $\mathcal{I} \subset \mathcal{O}_Z$ be the kernel of $\mathcal{O}_Z \to \Delta_* \mathcal{O}_X$. Define $\Omega_{X/Y} := \Delta^*(\mathcal{I}) = \mathcal{I} \otimes_{\mathcal{O}_Z} \mathcal{O}_X$.

Definition 4.1. Let $f : X \to Y$ be a morphism between adic spaces. Then f is called *unramified* if f is locally of finite type and if for any Huber pair (A, A^+) and any square-zero ideal $I \subset A$ and any morphism $\mathrm{Spa}(A, A^+) \to Y$, the map $\mathrm{Hom}_Y(\mathrm{Spa}(A, A^+), X) \to \mathrm{Hom}_Y(\mathrm{Spa}(A, A^+)/I, X)$ is injective. Say that f is *smooth* if it is locally of finite presentation and the above map is surjective. Say that f is *étale* if it is locally of finite presentation and the above map is bijective.

Remark 4.2. We need to clarify how to quotient a Huber pair (A, A^+) by an ideal I. The definition is simply that $(A, A^+)/J = (A/J, (A^+/(A^+ \cap J))^c)$ where $(A^+/(A^+ \cap J))^c$ denote the integral closure of $(A^+/(A^+ \cap J))^c$ inside A/J. This is again a Huber pair. This construction can be globalized to define closed adic subspaces of X corresponding to a quasicoherent $\mathcal{O}_X$-module $\mathcal{I} \subset \mathcal{O}_X$.

Remark 4.3. Open embeddings are étale, locally closed embeddings are unramified. For $f : X \to Y$ and $g : Y \to Z$, if $g \circ f$ is unramified then f is unramified. If $g \circ f$ is étale and g is unramified then f is étale. The above properties of morphisms are stable under base changes.

Remark 4.4. For $f : X \to Y$ locally of finite type, f is unramified if and only if $\Omega_{X/Y} = 0$, if and only if the diagonal $\Delta : X \to X \times_Y X$ is an open embedding. If $X \to Y$ is smooth then $\Omega_{X/Y}$ is a locally free $\mathcal{O}_X$-module.

Remark 4.5. A morphism $f : X \to Y$ is étale if and only if it is flat (for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$) and unramified.

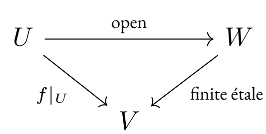
Proposition 4.6. *If* $\mathrm{Spa}(A, A^+) \to \mathrm{Spa}(B, B^+)$ *is étale, then* $B \to A$ *is flat.*

Proposition 4.7. *Every smooth morphism of adic spaces is open.*

Remark 4.8. There is a notion of étale and smooth morphisms between rigid analytic varieties and they coincide with the above definition under the functor r.

For analytic adic spaces, Scholze used the following criterion.

Proposition 4.9. *A morphism* $f : X \to Y$ *is* finite étale *if for all* $\mathrm{Spa}(B, B^+) \subset Y$ *open, its pullback in* X *is* $\mathrm{Spa}(A, A^+)$ *where* A *is a finite étale* B*-algebra and* A^+ *is the integral closure of the image of* B^+ *in* A*. Then a morphism* $f : X \to Y$ *is étale if and only if* $x \in X$ *there exists an open* $U \ni x$ *and* $V \supset f(U)$ *such that there is the following diagram.*



4.2. **The étale site.** The étale site $X_{\text{ét}}$ of an adic space is the category of adic spaces étale over X equipped with the Grothendieck topology such that a family a étale morphisms over X is a covering if and only if they are jointly surjective. Every morphism of adic spaces $f : X \to Y$ induces $f : X_{\text{ét}} \to Y_{\text{ét}}$. There is a parallel notion of the étale site over a rigid analytic variety. Then the functor r induces an equivalence on the étale topos on two incarnations of a rigid analytic variety X. Let's consider some examples of étale sheaves.

Example 4.10. Let Z be an adic space over X, then the presheaf on $X_{\text{ét}}$ represented by Z is a sheaf if either Z is étale over X or X is analytic.

Example 4.11. Let G be an étale finite adic group over X, then it makes sense to consider étale G-torsors over X. One may also view G as an étale sheaf, and that will lead to the same notion of étale torsors.

Example 4.12. There is a general recipe for taking base change of an adic space along a morphism between schemes. This basically amounts to analytify the scheme over a base adic space. One may in particular make sense of $Z := X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[T]/(T^n - 1)$ which is étale over X. The sheaf represented by Z is $\mu_n : Y \mapsto \{s \in \mathcal{O}_Y(Y) : s^n = 1\}$.

Example 4.13. The adic affine line $\mathbf{G}_a : Y \mapsto \mathcal{O}_Y(Y) = \mathrm{Hom}_X(Y, X \times_{\mathrm{Spec}\,\mathbf{Z}} \mathrm{Spec}\,\mathbf{Z}[T])$ is a sheaf. There is also the familiar sheaf $\mathbf{G}_m : U \mapsto \mathcal{O}_Y(U)^\times$ sitting in the exact sequence

$$0 \to \mu_n \to \mathbf{G}_m \xrightarrow{n} \mathbf{G}_m \to 0$$

5. Overconvergent sheaves

Huber studied the partially proper site which leads to the notion of overconvergent sheaves, which is very important for Bhargav's joint work with Jacob Lurie. Instead of considering all open embeddings, we might want to restrict to only partially proper open embeddings, and instead of étale morphisms, we might wish to restrict to partially proper étale morphisms. These notions lead to the partially proper sites on adic spaces.

Let X be an adic space. Then X as a topological space is locally spectral and generalizations of any given point form a chain. The morphisms between analytic spaces are spectral and generalizing. Recall that a map $X \to Y$ between locally spectral spaces is called spectral if for every qcqs open subsets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, the restriction $f : U \to V$ is quasi-compact.

Definition 5.1. A sheaf $\mathcal{F}$ on an analytic space X is called *overconvergent* if for any $x, y \in X$ such that y is a specialization of x, the natural mapping of stalks $\mathcal{F}_y \to \mathcal{F}_x$ is bijective.

Remark 5.2. Then $\mathcal{F}$ is overconvergent if and only if for every $x \in X$, the restriction of F to the set $\overline{\{x\}}$ of specializations of x in X is a constant sheaf.

Remark 5.3. Let $f : X \to Y$ be a morphism of analytic adic spaces. If $\mathcal{F}$ is overconvergent on Y, then $f^*\mathcal{F}$ is overconvergent.

Remark 5.4. Say that an open subset $U \subset X$ is partially proper if the inclusion is partially proper. Then $U \subset X$ is partially proper if and only if U is closed under specializations of X. The set of partially proper open subsets of X is closed under unions and finite intersections and hence defines a partially proper topology on X.

Definition 5.5. A sheaf $\mathcal{F}$ on the étale site $X_{\text{ét}}$ of an analytic adic space X is called *overconvergent* if for every specialization morphism $u : \eta_1 \to \eta_2$ of geometric points of X, the mapping $u^*\mathcal{F} : \mathcal{F}_{\eta_2} \to \mathcal{F}_{\eta_1}$ is bijective.