ADIC SPACES

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ABSTRACT. We give a rapid overview of the theory of adic spaces most relevant to Bhargav's class, topics including basic definitions, analytic adic spaces, rigid spaces, separated and proper morphisms, the generic fiber construction, the étale site, and overconvergent sheaves.

1. BASICS

The building blocks for schemes are spectra of rings. The building blocks for adic spaces are adic spectra of Huber pairs. One of the technical points of non-archimedean geometry is that all rings are equipped with a topology, and various constructions should take the topology into account. Let's start by reviewing what a Huber pair is.

Definition 1.1. A topological ring A is *Huber* if A admits an open subring $A_0 \subset A$ which is adic with respect to a finitely generated ideal of definition. Any such A_0 is called a ring of definition of A. By saying that a topological ring A_0 has I as an ideal of definition, we mean that the topology on A_0 is I-adic.

Remark 1.2. If A is Huber and $A_0 \subset A$ is any adic open subring, A_0 has a finitely generated ideal of definition.

Remark 1.3. A_0 is not assumed to be I-adically complete, but one may nonetheless always take the completion with respect to the I-adic topology by defining $\widehat{A}$ via the Cauchy sequence construction, then the closure $\widehat{A_0}$ of $A_0 \subset \widehat{A}$ is the I-adic completion of A_0 and $\widehat{A} = \widehat{A_0} \otimes_{A_0} A$. A Huber ring is called complete if it has an I-adically complete ring of definition (A_0, I). Whether a Huber ring is complete doesn't depend on the choice of a ring of definition. Although many of the below can be said for non-complete pairs, we always assume that A is complete.

Remark 1.4. When the ideal of definition is not assumed to be finitely generated, exotic things can happen. For example, there exists an I-adic ring A and an A-module M such that $\widehat{M}/I\widehat{M} \not\cong M/IM$. This is forbidden in p-adic geometry since one should be able to think of a p-adic formal scheme as patching other $\mathbf{Z}/p^n\mathbf{Z}$-schemes and hence one always wants completion to respect modulo p^n-reduction.

Example 1.5. Some recurring standard examples of Huber rings include:

(1) (Schemes) Any discrete ring A. Take $I = 0$, then any subring works as a ring of definition. Any subset of A is bounded. Hence, $A^\circ = A$ and $A^{\circ\circ} = \mathrm{Nil}(A)$. The pair (A, A) is a Huber pair. Spa(A, A) is the set of valuations on A bounded by 1. This is in general different from Spec A. For example the points of Spa$(\mathbf{Z}, \mathbf{Z})$ are:
   (a) A point η taking all non-zero integers to 1.
   (b) A special point s_p for each prime p given by the composition $\mathbf{Z} \to \mathbf{F}_p \to \{0, 1\}$ where the second map sends all nonzero elements to 1.
   (c) A point η_p for each prime p given by the p-adic valuation
   $$\mathbf{Z} \to \mathbf{Z}_p \to p^{\mathbf{Z}_{\leq 0}} \cup \{0\}.$$
   In general there is a natural map Spa$(R, R) \to$ Spec R and a natural section Spec $R \to$ Spa(R, R), both continuous.

(2) (Formal schemes) An adic ring A with a finitely generated ideal of definition is Huber, by taking $A_0 = A$. Any subset of A is bounded. Hence, $A^\circ = A$ and $A^{\circ\circ} = \sqrt{I}$. The pair (A, A) is a Huber pair. The map $A \mapsto$ Spa(A, A) defines a fully faithful functor from locally noetherian formal schemes to adic spaces.

(3) (Rigid spaces) Let k be a nonarchimedean field, then the Tate algebra

$$k\langle T_1,\dots,T_n\rangle := \left\{ \sum_{I\in \mathbf{Z}^n_{\geq 0}} a_I T^I \in k[[T_1,\dots,T_n]] : \lim_{|I|\to\infty} a_I \to 0 \right\}$$

is Huber with a ring of definition $\mathcal{O}_k\langle T_1,\dots,T_n\rangle$ and a ring of definition (g) for any $g\in\mathcal{O}_k$ with valuation less than 1. Quotients of Tate algebras with quotient topology are also Huber. A subset of A is bounded if the absolute values of the coefficients of the elements are uniformly bounded. Hence, $A^\circ = \mathcal{O}_k\langle T_1,\dots,T_n\rangle$ and $A^{\circ\circ} = \sqrt{(g)}$. The pair (A, A°) is Huber and the points of its adic spectrum can get complicated.

(4) (Perfectoid spaces) Let R be an integral perfectoid ring which is integrally closed in $R[1/p]$ (in particular R should be p-torsionfree). Then $R[1/p]$ is Huber with a ring of definition R and an ideal of definition (p).

Examples (3) and (4) are examples of Tate rings. Recall that a non-Archimedean field is a field k together with an equivalence class of non-Archimedean absolute values with respect to which k is complete. Also, recall that a non-Archimedean field is a local field if it is required to be locally compact Hausdorff, and in particular, that implies the residue field to be finite. Fix k a non-Archimedean field.

Definition 1.6. A Huber ring A is called *Tate* if it contains a topological nilpotent unit $g\in A$. Such a g is called a pseudo-uniformizer in A. A Huber ring A is *analytic* if the ideal generated by topologically nilpotent elements is the unit ideal. Any Tate ring is analytic.

A Huber pair (A, A^+) is a pair of a Huber ring A and a subring $A^+\subset A$ satisfying certain properties. Let's make this precise.

Definition 1.7. A subset S of a topological ring A is called *bounded* if for all open neighborhoods U of 0 there is an open neighborhood V of 0 such that $VS\subset U$.

Remark 1.8. A subring A_0 of a Huber ring A is a ring of definition if and only if it is open and bounded.

Definition 1.9. An element $x\in A$ of a Huber ring is *power-bounded* if $\{x^n : n\geq 0\}$ is bounded. Let $A^\circ\subset A$ be the subring of power-bounded elements. A Huber ring is *uniform* if A° is bounded, or equivalently, A° is a ring of definition.

Remark 1.10. Any ring of definition $A_0\subset A$ is contained in A°. Any two subrings of definition are contained in a third, and their union is A°.

Definition 1.11. Let A be a Huber ring. A subring $A^+\subset A$ is a *ring of integral elements* if it is open and integrally closed in A and $A^+\subset A^\circ$. A Huber pair is a pair (A, A^+) of a Huber ring A and A^+ a ring of integral elements.

Remark 1.12. One often takes $A^+ = A^\circ$, especially in cases corresponding to classical rigid geometry. The subset $A^{\circ\circ}\subset A$ of topologically nilpotent elements is always contained in A^+ by openness.

To these Huber pairs, we shall attach affine adic spaces, but we first need to define the corresponding "generalization" of prime ideals.

Definition 1.13. A *continuous valuation* on a topolological ring A is a map $|\cdot| : A\to\Gamma\cup\{0\}$ into a totally ordered abelian group Γ such that

(1) $|ab| = |a||b|$
(2) $|a+b|\leq \max(|a|,|b|)$
(3) $|1| = 1$
(4) $|0| = 0$
(5) for all $\gamma\in\Gamma$ lying in the image of $|\cdot|$, the set $\{a\in A : |a| < \gamma\}$ is open in A.

Two continous valuations $|\cdot|, |\cdot|'$ valued in Γ and Γ' are equivalent when $|a|\geq|b|$ if and only if $|a|' > |b|'$.

Definition 1.14. The adic spectrum Spa(A, A^+) is the set of equivalence classes of continuous valuations $|\cdot|$ on A such that $|A^+|\leq 1$. For $x\in$ Spa(A, A^+), write $g\mapsto |g(x)|$ for a choice of corresponding valuation. The topology on Spa(A, A^+) is generated by open subsets of the form

$$\{x : |f(x)|\leq |g(x)|\neq 0\}$$

with $f, g \in A$.

Remark 1.15. The topological space $\mathrm{Spa}(A, A^+)$ is spectral.

Remark 1.16. If $A \neq 0$, $\mathrm{Spa}(A, A^+)$ is nonempty. One has

$$A^+ = \{f \in A : f(x) \leq 1, \text{for all } x \in X\}$$

and an element $f \in A$ is invertible if and only if for all $x \in X$, we have $|f(x)| \neq 0$.

Definition 1.17. Let $s \in S$ and $T \subset A$ be a finite subset such that $TA \subset A$ is open. We define the subset

$$U(T/s) := \{x \in X : |t(x)| \leq |s(x)| \neq 0, \text{for all } t \in T\}.$$

Subsets of this form are called *rational subsets*. They are open since they are a finite intersection of $\{|t(x)| \leq |s(x)| \neq 0\}$ for $t \in T$.

Remark 1.18. The intersection of finitely many rational subsets is again rational.

Remark 1.19. It is a theorem of Huber for any rational subset $U \subset \mathrm{Spa}(A, A^+)$, there is a complete Huber pair $(A, A^+) \to (\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ such that the map

$$\mathrm{Spa}(\mathcal{O}_X(U), \mathcal{O}_X^+(U)) \to \mathrm{Spa}(A, A^+)$$

factors over U is universal for such maps. This map is in fact a homeomorphism onto U and implies that U is quasi-compact. Let's sketch how this is proved. Say $U = U(T/s)$. Let $A_0 \subset A$ be a ring of definition with I a finitely generated ideal of definition. Endow $A_0[t/s : t \in T]$ with the I-adic topology which extends to a ring topology on $A[1/s]$ making $A_0[t/s : t \in T]$ a ring of definition. Let $A[1/s]^+$ be the integral closure of $A^+[t/s : t \in T]$ in $A[1/s]$ and let $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ be the completion of $(A[1/s], A[1/s]^+)$.

Definition 1.20. Define a pair of presheaves $(\mathcal{O}_X, \mathcal{O}_X^+)$ of topological rings on $\mathrm{Spa}(A, A^+)$ by

$$\mathcal{O}_X(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X(U), \qquad \mathcal{O}_X^+(W) := \varprojlim_{U \subset W \text{ rational}} \mathcal{O}_X^+(U).$$

Remark 1.21. We may show that for all open U, we have

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1, \text{for all } x \in U\}.$$

In particular $\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is.

Definition 1.22. A Huber pair (A, A^+) is *sheafy* if $\mathcal{O}_X$ is a sheaf of topological rings.

Definition 1.23. We define the category of *locally v-ringed spaces* as follows. The objects are triples $(X, \mathcal{O}_X, |\cdot(x)|_{x \in X})$, where X is a topological space, $\mathcal{O}_X$ is a sheaf of topological rings, and for each $x \in X$, $|\cdot(x)|$ is an equivalence class of continuous valuations on $\mathcal{O}_{X,x}$. The morphisms are maps of topologically ringed topological spaces $f : X \to Y$ such that the composition

$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x} \xrightarrow{|\cdot(x)|} \Gamma_x$$

is equivalent to $|\cdot(y)|$. An *adic space* is a locally v-ringed space that is locally isomorphic to the adic spectrum of a sheafy Huber pair.

Remark 1.24. Huber pairs corresponding to adic noetherian rings with a finitely generated ideal of definition is sheafy. Discrete rings are sheafy. Huber pairs coming from quotients of Tate algebras and from perfectoid rings are sheafy.

Remark 1.25. The final object is $\mathrm{Spa}(\mathbf{Z}) = \mathrm{Spa}(\mathbf{Z}, \mathbf{Z})$.

Example 1.26. We already mentioned that classifying all points of an adic space corresponding to a rigid space can be complicated. We give such a complete classification for the adic closed unit disc $\mathbf{D}_K := \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ for an algebraically closed nonarchimedean field K. There are five classes of points:

(1) The classical points: Let $x \in \mathcal{O}_K$, then

$$K\langle T\rangle \to K \to \mathbf{R}_{\geq 0}, \qquad f \mapsto f(x) \mapsto |f(x)|$$

defines a continuous valuation.

(2) Branching points: Let $0 < r \leq 1$ be a real number and $x \in \mathcal{O}_K$, then

$$x_r : f \mapsto \sup_{y \in \mathcal{O}_K, |y-x| \leq r} |f(y)|$$

is a point of $\mathbf{D}_K$. It only depends on $D(x, r)$. When $r = 0$ is reduced to the classical points and when $r = 1$ this gives the Gauss point as the root of the tree. This class consists of the points where $r \in |K^\times|$

(3) Same as above but $r \notin |K^\times|$.

(4) Dead ends: Let $D_1 \supset D_2 \supset \cdots$ be a sequence of closed disks such that $\cap_i D_i = \emptyset$ (such sequences exists when $K = \mathbf{C}_p$). Then

$$f \mapsto \inf_i \sup_{y \in D_i} |f(y)|.$$

(5) Higher rank points: Let $x \in \mathcal{O}_K$ and fix a real number r with $0 < r \leq 1$. Endow the abelian group

$$\mathbf{R}_{>0} \times \gamma^{\mathbf{Z}}$$

with the unique total order such that $\gamma < r$ but $\gamma > r'$ for all $r' < r$. Then

$$x_{<r} : f = \sum_n a_n (t-x)^n \mapsto \max_n |a_n| \gamma^n$$

is a point of X. Similarly, one may define $x_{>r}$. If $r \notin |k^\times|$, $x_{<r} = x_{>r} = x_r$, but for each point x_r in type 2 this gives one extra point for each ray.