ON DERIVED COMPLETIONS AND OTHER THINGS

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This is a short expository note on derived completions and derived things, written mostly so I have an idea how my ReMpAHT talk will go. It might also be useful to people who attend the talk.

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1. What makes something derived?

When someone says 'animated X,' they mean that they have added extra information to X via 'simplicial methods.' The purpose of this talk is to say a little about what that means, at least in a handful of useful cases.

Let's start with a very basic instance of this phenomena. If X is a set, and $\sim$ is an equivalence relation on X, we can form the quotient set $X/\sim$.

For example, maybe $X = \mathbb{C}$ we use the equivalence relation $z \sim w$ if and only if there is some angle θ so that $z = we^{i\theta}$. Then the quotient set $X/\sim$ can be identified with the ray $[0, \infty)$.

This equivalence relation arose from the natural action of the group S^1 on X; our equivalence classes are just orbits of this group action. When we formed this quotient, we forgot one important piece of information: the orbit $\{0\}$ is unique, in that (unlike every other orbit), it has a large stabilizer. In other words, $0 \sim 0$ for 'many reasons,' whereas normally $z \sim w$ for exactly one reason.

One way to 'animate' this situation is to replace the quotient set $X/\sim$ by the pair of functions

$$s, t : X \times S^1 \to X,$$

where $s(z, \theta) = z$ (s stands for 'source') and $t(z, \theta) = ze^{i\theta}$ (t stands for 'target').

From this pair of functions s, t, we can recover the quotient set $X/\sim$; but knowing that our equivalence relation secretly came from an S^1 action gives us some extra information.

Earlier, I said that 'animated X' means adding extra information via 'simplicial methods.' What does that mean?

A *simplicial set* is just a combinatorial model of a shape; imagine a simplicial set as instructions telling you how many pieces of each dimension you have, as well as how to glue them together to build your shape. The precise definition is not so important for us, though.

This $s, t : X \times S^1 \to X$ can be thought of as behaving like a simplicial set.

Namely, if a *homotopy theorist* wanted to force two points of a space to be equal, they wouldn't quotient the space – they'd just add a path between the two points (since up to homotopy, you can contract that path to identify your two points). Thinking similarly, we will think of $X \times S^1$ as a set of 'paths' we adjoin to X; the two

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maps s, t being as they are tell us that $(z, \theta) \in X \times S^1$ is a path from $s(z, \theta) = z$ to $t(z, \theta) = ze^{i\theta}$. Thus we can view s, t as instructions for building a certain topological space, one where you start by taking the points of X with the discrete topology, and then connect them with paths – one path for each pair (z, θ).

It is perhaps surprising that this homotopy theory intuition works so well in algebra, but by a miracle it does. A point we will gloss over: there is an important theorem called the *Dold-Kan correspondence*, which says that (in a rather non-obvious way!), simplicial objects in an abelian category $\mathcal{A}$ are equivalent to (connective) chain complexes in $\mathcal{A}$. Think of the chain complex as being analogous to the chain complex one associates to a topological space: degree 0 is the free abelian group on the 0-simplices, degree 1 is the free abelian group on the 1-simplices, etc., and you have boundary maps between them.

People think of chain complex-y phenomena as being 'derived'; so 'animated' phenomena are the *connective* 'derived' phenomena. At least for Bhargav/Scholze/Lurie, as far as I am aware, so far derived schemes has always meant *animated* schemes – nobody in this school of works seems to have needed to do derived algebraic geometry for ring spectra which are not connective. This causes an unfortunate terminological problem where people say derived schemes when really they should be saying animated schemes; but anyways, I will follow the standard terminology, and in the next section we introduce *derived schemes*.

2. Derived intersections

Derived intersections are an interesting phenomenon, and we will need to understand them for derived completions anyway, so let's introduce them as our next example of a derived phenomena.

Take an affine scheme $X = \operatorname{Spec} A$. To intersect the two closed subschemes cut out by $f = 0$ and $g = 0$, we'd form $\operatorname{Spec} A/(f, g)$. Let's now form an 'animated' version of this quotient.

2.1. Derived quotient by a principal ideal.

Let's start by actually forming a derived version of $\operatorname{Spec} A/f$ first. The ring A/f is obtained by taking the ring A, and then quotienting by the equivalence relation $a \sim b$ if and only if $b - a = rf$ for some $r \in A$. Let's think of this r as the 'reason' that $a \sim b$. In particular, when f is a zero divisor, there can be 'multiple' reasons that $a \sim b$, just like we saw in our $X/\sim$ example above.

Example 2.1. Suppose $A = \mathbb{C}[x, y]/(xy)$; then $\operatorname{Spec}(A)$ is the union of the x-axis and the y-axis.

Setting $x = 0$ in the usual sense gives us the ring $A/x = \mathbb{C}[y]$, with spectrum just the x-axis. However, given any function $p(x, y) \in \mathbb{C}[x, y]/(xy)$, we have

$$p(x, y) \sim p(x, y)$$

for multiple reasons: once because

$$p(x, y) - p(x, y) = 0 \cdot x,$$

and a second time because

$$p(x, y) - p(x, y) = y \cdot x,$$

since in the ring A we have $y \cdot x = 0$. There also reasons like $3y, \pi y, iy$, etc.

Thus, the fact that x is a zero divisor menas that $p \sim p$ can happen for many distinct reasons!

In derived algebraic geometry, one replaces A/f by the chain complex $A/^{\mathbb{L}}f := A \xrightarrow{f} A$, with the first A being put in cohomological degree -1, and the second A being put in degree 0. This chain complex remembers A/f in the sense that

$$H^0(A \xrightarrow{f} A) = A/f,$$

but it also can remember the multiple reasons for $p \sim q$ to happen, in the form of knowing that

$$H^{-1}(A \xrightarrow{f} A) = \ker(f : A \to A).$$

I will remark on a point which will occur over and over again. Remembering the entire complex $A \xrightarrow{f} A$ is somehow 'tautological' – it remembers everything about the A/f situation, but everything is too much. When doing something derived, you will often have a very complicated model that remembers 'everything' – just like this complex – but then you'll care more about some other invariants associated to your model; these invariants are homotopy groups in the simplicial object view, or cohomology groups in the cochain complex view. Generally, π_0 or H^0 captures the 'naive' or 'underived' object, and the higher homotopy groups tell you more information. Think like a homotopy theorist: they care more about the 'homotopy type', and not really the actual topological space, when computing invariants; similarly, don't get too strung up on the 'model' of the derived object A/f, and instead just focus more on what invariants you can extract from it.[1]

2.2. **Derived intersection.** Now that we can form A/f in a derived sense, let's intersect $f = 0$ and $g = 0$ by forming $A/^{\mathbb{L}}(f, g)$ in a derived sense. In algebraic geometry, the 'standard' way we'd take an intersection would be a tensor product: we'd form $A/f \otimes_A A/g$.

In derived algebraic geometry, we will replace this by $A/^{\mathbb{L}}f \otimes^{\mathbb{L}}_A A/^{\mathbb{L}}g$. What does this mean? As a first step, we are going to replace A/f and A/g by there derived incarnations, the complexes

$$A \xrightarrow{f} A$$

and

$$A \xrightarrow{g} A.$$

Now, we form the tensor product *of these two complexes*. Then, we will see what extra information such a derived intersection remembers. It can be a little confusing to compute this tensor product of complexes because all the terms are A and $A \otimes_A A = A$, so when I write the final answer you will see A many times and not know 'which' A it is.

[1]Avalokiteśvara Bodhisattva, when practicing deeply the prajna paramita perceived that emptiness is form, and form is emptiness. This insight is very useful to homotopy theorists, and I think it helps me understand what is going on in homotopy theory.

2.2.1. *Tensor products of complexes.* So first let me tell you how to form the tensor product of

$$M^{-1} \xrightarrow{f} M^0$$

and

$$N^{-1} \xrightarrow{g} N^0.$$

The idea is to create a complex by doing some form of Dirichlet convolution to these two complexes; hopefully when I tell you what the answer is, you'll see why it's a reasonable notion of tensoring. The tensor product complex, which I will denote $M^\bullet \otimes_A N^\bullet$, lives in degrees $-2, -1$, and 0, and has terms

$$(M^\bullet \otimes_A N^\bullet)^{-2} = M^{-1} \otimes_A N^{-1},$$
$$(M^\bullet \otimes_A N^\bullet)^{-1} = (M^{-1} \otimes_A N^0) \oplus (M^0 \otimes_A N^{-1}),$$
$$(M^\bullet \otimes_A N^\bullet)^{0} = M^0 \otimes_A N^0,$$

The morphisms of the complex are as follows.

Warning 2.2. There will be some funny signs in the differentials defining this complex. In a moment I will say why one might want to use these fun signs; sometimes people refer to the correct way to ascribe signs as the 'Koszul sign rule.' They usually use this term in a very general context; whenever you see chain complexes with signs, the author will tell you they used the 'Koszul sign rule' to figure out where the signs go. The 'Koszul sign rule' is just that, whenever you need to 'move the differential' past a term of degree n, you add the sign $(-1)^n$. So for example, writing m^k to denote an element of degree k, if you do somethign like

$$d(m^k \otimes n^j),$$

then you should apply the Leibniz rule with proper signs to get

$$d(m^k \otimes n^j) := d(m^k) \otimes n^j + (-1)^k m^k \otimes d(n^j).$$

If you've ever taken the exterior derivative of differential forms, you see exactly the same sorts of signs showing up.

When going from degree -2 to degree -1, we use the map

$$m^{-1} \otimes n^{-1} \mapsto f(m^{-1}) \otimes n^{-1} - m^{-1} \otimes g(n^{-1}).$$

Do you see why that answer lives in $(M^{-1} \otimes N^0) \oplus (M^0 \otimes N^{-1})$?

When going from degree -1 to degree 0, we use the maps

$$m^{-1} \otimes n^0 \mapsto f(m^{-1}) \otimes n^0,$$
$$m^0 \otimes n^{-1} \mapsto m^0 \otimes g(n^{-1}).$$

Remark 2.3. Why use these funny signs? The trouble is that, if you made everything positive, the differentials in this chain complex would not obey $d^2 = 0$. However, with our carefully chosen signs, we will have $d^2 = 0$. Let's compute this explicitly:

$$\begin{aligned} d(d(m^{-1} \otimes n^{-1})) &= d(f(m^{-1}) \otimes n^{-1} - m^{-1} \otimes g(n^{-1})) \\ &= d(f(m^{-1}) \otimes n^{-1}) - d(m^{-1} \otimes g(n^{-1})) \\ &= f(m^{-1}) \otimes g(n^{-1}) - f(m^{-1}) \otimes g(n^{-1}) \\ &= 0. \end{aligned}$$

Note that if we had a + sign in our map from degree −2 to degree −1, then this would not be zero!

2.2.2. *Back to our derived intersection.* If we apply that above notion of tensor product to our origianl complexes $A \xrightarrow{f} A$ and $A \xrightarrow{g} A$, then we get

$$A/f \otimes_A^{\mathbb{L}} A/g = A \xrightarrow{(f,-g)} A \oplus A \xrightarrow{(g,f)} A.$$

Here, I mean the tensor product is (modelled by!) a complex living in degrees $-2, -1$, and 0. The differential from degree -2 to degree -1 is

$$a \mapsto (fa, -ga),$$

and the differential from degree -1 to degree 0 is

$$(a_1, a_2) \mapsto ga_1 + fa_2.$$

Note that the composition is zero, since

$$a \mapsto (fa, -ga) \mapsto g(fa) + f(-ga) = (fg)a - (fg)a = 0.$$

Remark 2.4. This type of three term complex is called a *Koszul complex* associated to the sequence f, g.

Remark 2.5 (Technical thing for experts)**.** If f is not a zero divisor, then

$$A \xrightarrow{f} A$$

is a free resolution of A/f, and so we can compute the derived tensor product as just

$$A/g \otimes_A (A \xrightarrow{f} A) = A/g \xrightarrow{f} A/g.$$

When f is a zero divisor, though, this will in general be incorrect – for example, consider the H^{-2} of that complex vs the Koszul complex.

This complex has three terms, and so its entitled to have cohomologies in degrees $-2, -1$, and 0. Let's compute these. The easiest one is

$$H^0(A/^{\mathbb{L}}f \otimes_A^{\mathbb{L}} A/^{\mathbb{L}}g) = A/(f, g).$$

This is expected: H^0 of derived X really ought to be the classical X!

Next,

$$H^{-1}(A/^{\mathbb{L}}f \otimes_A^{\mathbb{L}} A/^{\mathbb{L}}g) = \{(a_1, a_2) \in A^2 \mid ga_1 + fa_2 = 0\}/\{(fa, -ga) \mid a \in A\}.$$

Finally,

$$H^{-2}(A/^{\mathbb{L}}f \otimes_A^{\mathbb{L}} A/^{\mathbb{L}}g) = \{a \in A \mid fa = ga = 0\}.$$

So, this derived intersection has two extra pieces of information.

2.3. **What is this information good for?** It turns out that this extra information is exactly what one should encode for intersection theory to work correctly without needing to add weasel words like 'transverse intersections' etc. in things like Bezout's theorem (although already underived schemes can understand all Bezout phenomena in $\mathbb{P}^2$; really its only in higher $\mathbb{P}^n$ that derived algebraic geometry can help).

To give a very simple example, consider the statement 'every two lines in $\mathbb{P}^2$ intersect exactly once.' This is actually false: if the two lines are *equal*, they intersect too often. Let me explain now how to take the intersection of a line with itself in derived algebraic geometry. To make the computation simpler, let's work actually in $\mathbb{A}^2 = \operatorname{Spec}\mathbb{C}[x,y]$, with the line being $x=0$. Then one can compute the 'derived' intersection $x=0, x=0$ is

$$\mathbb{C}[x,y] \xrightarrow{(x,-x)} \mathbb{C}[x,y] \oplus \mathbb{C}[x,y] \xrightarrow{(x,x)} \mathbb{C}[x,y].$$

More importantly, the cohomologies of this complex are

$$H^0 = \mathbb{C}[x,y]/(x),$$
$$H^{-1} = (1,-1)\mathbb{C}[x,y]/(x,-x)\mathbb{C}[x,y] = \mathbb{C}[x,y]/(x),$$
$$H^{-2} = 0.$$

In fact, this complex is even quasiisomorphic to

$$\mathbb{C}[y] \oplus (\mathbb{C}[y])[-1],$$

but maybe I won't say too much about what that means.

What does knowing this extra H^{-1} give us? In just this one affine chart it's hard to see, but let's look at another chart of $\mathbb{P}^2$ to try and globalize. If we view $\mathbb{P}^2$ as having coordinates $[X:Y:Z]$, then before we were working in the chart $Z \neq 0$, with coordinates $x = X/Z, y = Y/Z$. In these global coordinates, our line is the set of all points $[0:Y:Z]$, and hence is a copy of $\mathbb{P}^1$. Our chart $\mathbb{A}^2$ missed one point of our line: the point $[0:1:0]$.

We could have instead worked in the chart $Y \neq 0$, with coordinates $x' = X/Y, z' = Z/Y$. Now our line becomes $x' = 0$, and the point $[0:1:0]$ is visible to our chart. The derived self intersection works out to be $\mathbb{C}[z'] \oplus (\mathbb{C}[z'])[-1]$ for the same reason as before.

The overlap of our two charts is given by $\mathbb{C}[x,y,1/y]$, with

$$x' = \frac{x}{y},$$
$$z' = \frac{1}{y}.$$

Hence, the *global* H^{-1} is the twisting sheaf $\mathcal{O}(-1)$ on $\mathbb{P}^2$. Note that this twisting sheaf $\mathcal{O}(-1)$ is exactly the normal bundle of a line in $\mathbb{P}^2$ – this is not a coincidence!

3. Back to classical algebra: The topological approach to completions

A miracle in p-adic analysis is that convergence of infinite series becomes a completely algebraic phenomena. Indeed, take K a complete non-archimedean field with valuation ring $\mathcal{O}_K$. Then a series $\sum_{n\geq 0} a_n$ of terms in $\mathcal{O}_K$ converges if and only if the terms a_n go to 0, which happens if and only if all but finitely many a_n lie in $\mathfrak{m}_K^N$ for each N.

This motivates the following general notion.

Lemma 3.1. *Let A be a ring and I an ideal. The sets*

$$a + I^n$$

form the basis of a topology on A. Moreover, they form a 'non-archimedean basis': the intersection of two such sets is either empty, or equal to one of the sets.

We call this the I-adic topology *on A.*

Proof. Let

$$X = (a + I^n) \cap (b + I^{n+m}).$$

If $x \in X$, then we can write

$$x = a + i_n = b + i_{n+m}.$$

Hence $a - b = i_{n+m} - i_n \in I^n$.

In that case, $a + I^n = b + I^n$, and so $X = b + I^{n+m}$. Thus either $X = \emptyset$ or $X = b + I^{n+m}$, and in either way the intersection of two sets in our base lies in our base. □

Definition 3.2. We say that (A, I) is *separated* if the I-adic topology on A is Hausdorff; equivalently, if $\bigcap_{n=1}^{\infty} I^n = 0$.

Remark 3.3. As A is a topological ring, Hausdorfness is equivalent to being T_0 (points are topologically distinguishable) is equivalent to the identity being topologically distinguishable from every non-identity point; as $\bigcap_n I^n$ is precisely the intersection of all open sets containing 0, we get the equivalence between Hausdorfness and the intersection.

Proposition 3.4. *Define*

$$\nu : A \to \mathbb{N} \cup \{\infty\}$$

the function

$$\nu(0) = \infty,$$

$$\nu(a) = \max\{n \in \mathbb{N} \mid a \in I^n\}.$$

Then $d(x, y) = 2^{-\nu(x-y)}$ is a pseudometric on A inducing the I-adic topology. If (A, I) is separated, then d is a metric.

Proof. Immediate from the same algebra used to prove that the p-adic metric is a metric. □

Thus we can view A as a pseudometric space with the I-adic topology, not just a metric space.

Definition 3.5. We say A is *I-adically complete* if (A, I) is separated, and A is complete with respect to the I-adic pseudometric.

4. The algebraic approach to completions

We now give a more algebraic criterion for a ring A to be I-adically complete. This is the miracle of the non-archimedean world: there is a completely algebraic way to understand convergence!

Theorem 4.1. *Let A be a ring and I an ideal. The natural map*

$$A \to \varprojlim_n A/I^n$$

is

(1) injective *if and only if (A, I) is separated,*
(2) bijective *if and only if A is I-adically complete.*

Proof. Call this map φ. Note that $\ker\varphi = \bigcap_{n\geq 0} I^n$ and so we get the injectivity claim immediately.

Now assume φ is injective; let's show under this assumption, surjectivity is equivalent to completeness.

If A is complete, we have surjectivity as follows: given $(\bar{a_0}, \bar{a_1}, ...) \in \varprojlim_n A/I^n$, let a_i be an arbitrary lift of $\bar{a_i} \in A/I^{i+1}$ to A, and then define

$$a := a_0 + (a_1 - a_0) + (a_1 - a_2) + \cdots ,$$

which exists by I-adic completeness and of course maps to $(\bar{a_0}, \bar{a_1}, ...)$ in the inverse limit.

Conversely, if φ is surjective, then we can use φ^{-1} to form infinite sums and hence get I-adic completion. □

This inverse limit approach is very algebraically convenient. It also allows us to define a completion functor.

Definition 4.2. Let A be a ring. We define the *completion of A with respect to an ideal I* to be

$$A_I^\wedge := \varprojlim_n A/I^n.$$

There is a natural map $A \to A_I^\wedge$.

Warning 4.3. $A_I^\wedge$ is not in general complete!

Proposition 4.4. *Take A a ring, I an ideal; let $\varphi : A \to A_I^\wedge$ be the natural map. For J an ideal of A, write $JA_I^\wedge$ to mean $\varphi(J)A_I^\wedge$.*

Then

$$I^n A_I^\wedge = (IA_I^\wedge)^n$$

for every n, and the projection

$$A_I^\wedge \to A/I^n$$

is surjective with kernel containing $I^n A_I^\wedge$.

Moreover, if I is finitely generated*, then the kernel is exactly $I^n A_I^\wedge$; in particular, $A_I^\wedge$ is I-adically complete in this case.*

Remark 4.5. This funny business with finite generation is our first clue that completion is not quite the right notion, and we need something derived. We will explore this soon.

5. Crash course in derived categories

By the Dold-Kan correspondence mentioned above, simplicial objects of an abelian category can be replaced by connective chain complexes; the Dold-Kan correspondence also takes the homotopy group functors to cohomology functors, and so the correct analogue of 'look at simplicial objects, but only up to weak homotopy equivalences' becomes 'look at connective chain complexes, but only up to quasi-isomorphisms'. Here, a *quasi-isomorphism* is a map which is an isomorphism on all cohomologies.

Of course, there is no reason to restrict to connective chain complexes, and so we don't. This lets us define the *derived category* of an abelian category $\mathcal{A}$ as the new category $\mathcal{D}(\mathcal{A})$, whose objects are 'chain complexes in $\mathcal{A}$ up to quasi-isomorphism.' I won't say too much about what this means, but I will say the main feature of this category that we use in practice; in our derived completion example, we will use this a lot, so hopefully you'll get comfortable working with $\mathcal{D}(\mathcal{A})$ from that.

In any abelian category, the most important constructions are kernels and cokernels. In $\mathcal{D}(\mathcal{A})$, these operations magically morph into one, which we call the *mapping cone*. The mapping cone measures how far a morphism $f : X^\bullet \to Y^\bullet$ is from being a quasi-isomorphism, by measuring both the kernel and the cokernel of the maps $H^i(X^\bullet) \to H^i(Y^\bullet)$.

More precisely, we have the following.

Theorem 5.1. *For any morphism $f : X^\bullet \to Y^\bullet$ of complexes, there is a complex $C(f)$ and morphisms*

$$X^\bullet \to Y^\bullet \to C(f) \to X^\bullet[1]$$

(where the $[1]$ means 'take the same complex, but shift all the terms one to the left; note that $H^q(X^\bullet[1]) = H^{q+1}(X^\bullet)$) such that the associated sequences

$$H^q(X^\bullet) \to H^q(Y^\bullet) \to H^q(C(f)) \to H^{q+1}(X^\bullet)$$

on cohomologies are all exact; concatenating them, we get a long exact sequence

$$\cdots \to H^q(X^\bullet) \to H^q(Y^\bullet) \to H^q(C(f)) \to H^{q+1}(X^\bullet) \to \cdots .$$

People refer to situations

$$X^\bullet \to Y^\bullet \to C(f) \to X^\bullet[1]$$

as *exact triangles*, and a typical usage would be hearing someone say

$$X \to Y \to Z$$

is an exact triangle; this just means that they can identify Z with the mapping cone of $X \to Y$.

At the dawn of time, Grothendieck and Verdier gave us a 1-category $\mathcal{D}(\mathcal{A})$ in which mapping cones behaved very poory functorially. Lurie has given us the *stable ∞-category*, which is some sort of category-like object $\mathcal{D}(\mathcal{A})$ in which forming cones is genuinely functorial... for a different sense of functorial. In practice, the language people use when referring to stable ∞-categories and ordinary derived categories is usually identical, so it won't matter so much that below I write in the language of stable ∞-categories; hopefully you'll pick up how the language is used. The key difference is that, because mapping cones are functorial to me, I can slightly simplify some technical arguments that would require more care if you wanted to do everything using the classical language.[2]

6. Derived completion

We now discuss the theory of derived completion. The purpose of derived completion is to correct all the annoying finiteness assumptions needed for usual completion to be exact. Another consequence is that it will let us study completion of *complexes*, as opposed to individual modules; this allows us to get a notion of completion for objects of the derived category.

There are, perhaps unfortunately, many distinct notions of derived completion. We will use one inspired by derived algebraic geometry. To be precise, we are

[2]There is a conservation of difficulty in mathematics; the only reason I can give simpler arguments is because Lurie wrote two books called *Higher Topos Theory* and *Higher Algebra*, in which he gave broadly applicable lemmas basically recreating every technical argument one usually gives with derived categories.

going to set up a theory of derived completions which is well adapted to formal schemes; this means we will take an ambient ring A, and an *explicit list* of elements $f_1, ..., f_r \in A$ generaitng an ideal $I = (f_1, ..., f_r)$. We will then form the derived formal scheme associated to this setup.

This variant of derived completion is the one which will arise in Bhargav's course. Why? Well, last week Dhruv introduced formal schemes, and explained how completions are intimiately related to them. The completions Bhargav will be considering will have this flavor, of coming from a formal scheme/deformation theory setup. Thus we use this variant of derived completion, motivated by just taking the usual derived completion but in the sense of derived algebraic geometry.

We now define derived completions. We will give several equivalent notions of derived completion, each having several advantages. Our first is the most intuitive from our above discussion; fix A a ring, and $f_1, ..., f_r \in A$. Set $I = (f_1, ..., f_r)$. The classical I-adic completion of $M \in \mathrm{Mod}_A$ with respect to this ideal I would be

$$\varprojlim_n M/I^n M.$$

We are going to change this in three ways:

(1) allow M to live in $\mathcal{D}(A)$,
(2) replace $M/I^n M$ with a derived quotient,
(3) replace $\varprojlim_n$ with $R\varprojlim_n$.

The hardest of these to fix is quotienting by I^n; the trouble is that derived quotients are sensitive to the choice of generating set (on purpose!), and so we need to pick a generating set of I^n. To avoid having to do that, we will just derived quotient by $f_1^n, f_2^n, ..., f_r^n$; the idea here is that, in the classical setting, these quotients would be final in the original system and so the limit would change, and in the derived situation this is easier.

Definition 6.1. We define the $(f_1, ..., f_r)$-adic completion of $M \in \mathcal{D}(A)$ (which is sensitive to our presentation of the ideal!) by

$$\hat{M} = R\varprojlim_n (M \otimes^{\mathbb{L}}_{A[x_1,...,x_r]} A[x_1, ..., x_r]/(x_1^n, ..., x_r^n),$$

where we view M as an $A[x_1, ..., x_r]$-module by having x_i act as f_i, and implicitly we forget the x_i actions on $\hat{M}$ to view it as an element of $\mathcal{D}(A)$.

Definition 6.2. We say that $M \in \mathcal{D}(A)$ is *derived* $(f_1, ..., f_r)$*-complete* if the natural map

$$M \to \hat{M}$$

is an equivalence.

For understanding derived completion, note that we always have a natural map $M \to \hat{M}$. Understanding the cone of this map can help us to understand derived completions.

Lemma 6.3. *Let $M \in \mathcal{D}(A)$, and $f_1, ..., f_r \in A$. Define a complex $Q \in \mathcal{D}(A)$ by*

$$Q = \left(A \to \prod_{i=1}^{r} A_{f_i} \to \prod_{i<j} A_{f_i f_j} \to \cdots \to A_{f_1 \cdots f_r} \right),$$

where A is placed in degree 0.

Then $\hat{M} = R\operatorname{Hom}_A(Q, M)$.

Proof. The quotient $A[x_1, ..., x_r]/(x_1^n, ..., x_r^n)$ is resolved by the Koszul complex

$$K(f_1^n, ..., f_r^n) := A \to \prod_{i=1}^{r} A \to \prod_{i<j} A \to \cdots \to A.$$

Here, we use that $x_1^n, ..., x_r^n$ still form a regular sequence in $A[x_1, ..., x_r]$, and the A on the *left* is placed in degree $-r$.

Hence $M \otimes^{\mathbb{L}} A[x_1, ..., x_r]/(x_1^n, ..., x_r^n)$ is equal to

Recalling filtered colimits are exact, and so there's no derived funny business, we have

$$A_f = \operatorname{colim}(A \xrightarrow{f} A \xrightarrow{f} \cdots).$$

By applying this termwise, we deduce that

$$\operatorname{colim}_n K(f_1^n, ..., f_r^n) = Q[r].$$

The shift by r is because the Koszul complex started in degree $-r$, but Q started in degree 0.

Therefore

$$\begin{aligned} R\operatorname{Hom}_A(Q, M) &= R\operatorname{Hom}_A(\operatorname{colim}_n K(f_1^n, ..., f_r^n)[-r], M) \\ &= R\varprojlim_n R\operatorname{Hom}_A(K(f_1^n, ..., f_r^n)[-r], M). \end{aligned}$$

But famously, the Koszul complex is perfect, with dual $K(f_1^n, ..., f_r^n)[-r]$. In other words,

$$R\operatorname{Hom}_A(Q, M) = R\varprojlim_n M \otimes_A^{\mathbb{L}} K(f_1^n, ..., f_r^n) = R\varprojlim_n M \otimes_A^{\mathbb{L}} A[x_1, ..., x_r]/(x_1^n, ..., x_r^n) = \hat{M},$$

exactly as desired. □

The purpose of Lemma 6.3 is that now the map $M \to \hat{M}$ is simply the map induced by $Q \to A$ which is identity in degree 0 (as A is concentrated in one degree, this map will be a morphism of chain complexes). Hence the cone of $M \to \hat{M}$ is $R\operatorname{Hom}_A(\operatorname{fib}(Q \to A), M)$, which gives us a universal description of the cone of $M \to \hat{M}$.

Theorem 6.4. *Let* $f_1, ..., f_r \in A, M \in \mathcal{D}(A)$. *Set* $I = (f_1, ..., f_r)$. *The following conditions are equivalent:*

(1) M *is derived* $(f_1, ..., f_r)$*-complete;*
(2) $R\operatorname{Hom}_A(A_{f_i}, M) = 0$ *for each* i*;*
(3) $R\operatorname{Hom}_A(A_f, M) = 0$ *for each* $f \in I$.

Moreover, $\hat{M}$ *is always derived* $(f_1, ..., f_r)$*-complete.*

Remark 6.5. In particular, being *complete* is independent of our choice of generating set for the ideal; so we now can just say derived I-complete.

Proof. First, we prove that $R\operatorname{Hom}_A(A_{f_n}, \hat{M}) = 0$ for each f_n. By Lemma 6.3, it suffices to prove $A_{f_n} \otimes_A^{\mathbb{L}} Q = 0$.

As A_g is always a flat A-module, we can take this tensor product naively to compute it as

$$A_{f_n} \to \prod_{i=1}^{r} A_{f_n f_i} \to \cdots \to A_{f_n f_1 \cdots f_r}.$$

This is just the Koszul complex of $f_1, ..., f_r$ in A_{f_n}; but the Koszul complex associated to any sequence containing a unit is quasi-isomorphic to 0 (see stacks 0G6J for an explicit homotopy given) and hence this vanishes.

It follows that, if M is derived $(f_1, ..., f_r)$-complete, then $R\operatorname{Hom}_A(A_{f_i}, M) = 0$ for each i (since in this case $M = \hat{M}$).

Next, we prove that the set of all f such that $R\operatorname{Hom}_A(A_f, M) = 0$ forms an ideal of A (if we say formally declare $A_0 = 0$, so that $f = 0$ always lies in this set); this shows 2 implies 3.

And indeed,

$$R\operatorname{Hom}_A(A_{\lambda f}, M) = R\operatorname{Hom}_A(A_\lambda, R\operatorname{Hom}_A(A_f, M))$$

by the tensor-hom adjunction, which implies the set of all such f is closed under multiplication.

If f, g are such that $f + g \neq 0$ and $R\operatorname{Hom}_A(A_f, M) = R\operatorname{Hom}_A(A_g, M) = 0$, then $R\operatorname{Hom}_A(A_{f+g}, M) = 0$ as well, since there is a short exact sequence

$$0 \to A_{f+g} \to A_{f(f+g)} \oplus A_{g(f+g)} \to A_{fg(f+g)} \to 0.$$

It follows that there is an exact triangle

$$R\operatorname{Hom}_A(A_{f+g}, M) \to R\operatorname{Hom}_A(A_{f(f+g)}, M) \oplus R\operatorname{Hom}_A(A_{g(f+g)}, M) \to R\operatorname{Hom}_A(A_{fg(f+g)}, M).$$

But it's easy to see all the terms of this triangle except $R\operatorname{Hom}_A(A_{f+g}, M)$ are zero, forcing this last term to be zero as well. We conclude.

Finally, we must prove that 3 implies 1. To do this, we study the cone of $M \to \hat{M}$, and express it using terms $R\operatorname{Hom}_A(A_f, M)$. This cone, recall, is just $R\operatorname{Hom}_A(\operatorname{fib}(Q \to A), M)$. The key point is that $\operatorname{fib}(Q \to A)$ can be explicitly represented by a bounded complex whose terms involve only things of the form A_f for some $f \in I$.

More precisely, by exactness of $R\operatorname{Hom}_A(-, M)$, the class of complexes $N \in \mathcal{D}(A)$ for which $R\operatorname{Hom}_A(N, M) = 0$ is stable under extensions. The class includes all the A_f, and hence includes any bounded chain complex made of the A_f.

Our argument that 1 implies 2 really showed that any $\hat{M}$ obeyed condition 2, which (now that we know 2 implies 3 implies 1) proves $\hat{M}$ is derived complete, as desired. □

There is also a derived version of our tensor product formula.

Proposition 6.6. *Let $M \in \mathcal{D}(A)$, and $f \in A$. Then $\hat{M}$ is derived complete, and in fact*

$$\hat{M} = M \otimes_A^{\mathbb{L}} \hat{A}.$$

In particular, $\hat{A}$ is a derived idempotent A-algebra.

Proof. This follows from the remarkable fact that fibers can also be viewed as cofibers. Namely, instead of defining $\hat{M}$ as the fiber of a map, we could also define it as the cofiber of

$$R\operatorname{Hom}_A(\operatorname{cofib}(Q \to A), M) \to R\operatorname{Hom}_A(A, M) = M.$$

Set $C = \operatorname{cofib}(Q \to A)$. Then

$$\hat{A} = \operatorname{cofib}(R\operatorname{Hom}_A(C, M) \to M).$$

It follows from cocontinuity of the tensor product that

$$\begin{aligned} M \otimes^{\mathbb{L}}_A \hat{A} &= M \otimes^{\mathbb{L}}_A \operatorname{cofib}(R\operatorname{Hom}_A(C, A) \to A) \\ &= \operatorname{cofib}((M \otimes^{\mathbb{L}}_A R\operatorname{Hom}_A(C, A)) \to M) \\ &= \operatorname{cofib}(R\operatorname{Hom}_A(C, M) \to M) \\ &= \hat{M}. \end{aligned}$$

□

Remark 6.7. Set $A = k[x]$. Let's compute the derived completion of A with respect to (x, x). It's easiest to do this using the $A \to A_x \oplus A_x \to A_{x^2}$ complex approach, but because I think it's more instructive as to why the weird derived double intersection phenomena disappears in the inverse limit, we will take a slightly different approach.

Set K_n the Koszul complex

$$A \xrightarrow{(-x^n, x^n)} A \oplus A \xrightarrow{(x^n, x^n)} A,$$

which we treat as spanning degrees -2 to 0. The cohomologies are

$$H^0(K_n) = k[x]/(x^n),$$

$$H^{-1}(K_n) = k[x]/(x^n),$$

$$H^{-2}(K_n) = 0.$$

The inverse system has the connecting maps $K_{n+1} \to K_n$ defined by

$$\begin{tikzcd} A \arrow[r, "(-x^{n+1}\ x^{n+1})"] \arrow[d, "\mathrm{id}"] & A \oplus A \arrow[r, "(x^{n+1}\ x^{n+1})"] \arrow[d, "x"] & A \arrow[d, "\mathrm{id}"] \\ A \arrow[r, "(-x^n\ x^n)"] & A \oplus A \arrow[r, "(x^n\ x^n)"] & A. \end{tikzcd}$$

We wish to compute $R\varprojlim_n K_n$. Let's start by computing the cohomologies. The Milnor sequence reads

$$0 \to R^1 \varprojlim_n H^{q-1}(K_n) \to H^q(R\varprojlim_n K_n) \to \varprojlim_n H^q(K_n) \to 0.$$

These K_n each only have cohomologies in degrees 0 and -1. Therefore unless $q \in \{0, -1, 1\}$, the cohomology $H^q(R\varprojlim_n K_n)$ is forced to vanish. For $q = -1$, since $H^{-2}(K_n) = 0$, we get an isomorphism

$$H^{-1}(R\varprojlim_n K_n) = \varprojlim_n H^{-1}(K_n).$$

Our connecting homomorphisms in the inverse system induce maps

$$H^{-1}(K_{n+1}) \to H^{-1}(K_n)$$

of the form

$$k[x]/(x^{n+1}) \xrightarrow{x} k[x]/(x^n).$$

If we just used the obvious projections, the inverse limit would be $k[[x]]$. But because we use these multiplication by x projections, the inverse limit is actually zero! Thus in the limit, all of our derived intersection phenomena went away.

7. A few important properties of derived completion

We now study a few important properties of derived I-completion.

Fix I a *finitely generated* ideal of a ring A.

Definition 7.1. Let

$$\mathcal{D}_{I-comp}(A)$$

denote the full subcategory of $\mathcal{D}(A)$ spanned by the derived I-complete objects.

Proposition 7.2. *Fix a generating set $f_1, ..., f_r \in A$ of the ideal I. Then the functor*

$$M \mapsto \hat{M}$$

sending $M \in \mathcal{D}(A)$ to its completion with respect to this generating set is left adjoint *to the inclusion*

$$\mathcal{D}_{I-comp}(A) \hookrightarrow \mathcal{D}(A).$$

In particular, this completion is actually independent of our choice of generating set, by uniqueness of adjoints.

Remark 7.3. Here, we mean adjunction at the level of ∞-categories: that is, we have an equivalence

$$R\operatorname{Hom}_A(\hat{M}, N) = R\operatorname{Hom}_A(M, N)$$

whenever $N \in \mathcal{D}_{I-comp}(A)$ and $M \in \mathcal{D}(A)$.

Remark 7.4. In particular, the inclusion

$$\mathcal{D}_{I-comp}(A) \hookrightarrow \mathcal{D}(A)$$

is a localization.

Proof. Consider the exact triangle

$$A \to \hat{A} \to Q,$$

with Q being defined as the cone of $A \to \hat{A}$. Recall from our proof of **??** that Q has the property that $R\operatorname{Hom}_A(Q, N) = 0$ for all derived I-complete N.

We therefore get an exact triangle

$$M \to \hat{M} = M \otimes_A^{\mathbb{L}} \hat{A} \to M \otimes_A^{\mathbb{L}} Q,$$

which induces a canonical exact triangle

$$R\operatorname{Hom}_A(M \otimes_A^{\mathbb{L}} Q, N) \to R\operatorname{Hom}_A(\hat{M}, N) \to R\operatorname{Hom}_A(M, N)$$

for any N. If N is derived I-complete, then the tensor-hom adjunction implies the first term of this triangle vanishes, and hence we have an equivalence $R\operatorname{Hom}_A(\hat{M}, N) = R\operatorname{Hom}_A(M, N)$, natural in M, N. Thus we have an adjunction. □

Proposition 7.5 (Serre subcategory property)**.** *The subcategory $\mathcal{D}_{I-comp}(A)$ of $\mathcal{D}(A)$ is a stable ∞-category. More precisely, if*

$$X \to Y \to Z$$

is an exact triangle in $\mathcal{D}(A)$, and two out of three terms lie in $\mathcal{D}_{I-comp}(A)$, then so does the third. Thus fibers and cofibers are formed in the expected way, and derived I-complete objects are stable under extensions.

Remark 7.6 (Stable snake lemma)**.** In the below, we shall use the snake lemma in stable ∞-categories. Let me say a few words on it; in the triangulated context, this is called the 3×3 lemma, and is much worse: the statement already is quite subtle and the proof uses the octahedral axiom repeatedly.

Take $\mathcal{C}$ a stable ∞-category, and let

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r] & Y' \arrow[r] & Z' \end{tikzcd}$$

be a commutative diagram in $\mathcal{C}$ with exact rows. It follows that

$$Z = \mathrm{cofib}(X \to Y),$$
$$Z' = \mathrm{cofib}(X' \to Y'),$$

and h is uniquely determined by the universal property of colimits.

Functoriality of colimits gives us natural maps

$$\mathrm{cofib}(X \to X') \to \mathrm{cofib}(Y \to Y') \to \mathrm{cofib}(Z \to Z'),$$

assembling into a 3×3 commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] & Y \arrow[r] \arrow[d, "g"] & Z \arrow[d, "h"] \\ X' \arrow[r] \arrow[d] & Y' \arrow[r] \arrow[d] & Z' \arrow[d] \\ \mathrm{cofib}(f) \arrow[r] & \mathrm{cofib}(g) \arrow[r] & \mathrm{cofib}(h). \end{tikzcd}$$

We claim now that the bottom row of this commutative diagram is also exact. Indeed, colimits commute with colimits, so

$$\mathrm{cofib}(h) = \mathrm{cofib}(\mathrm{cofib}(X \to Y) \to \mathrm{cofib}(X' \to Y')) = \mathrm{cofib}(\mathrm{cofib}(f) \to \mathrm{cofib}(g)).$$

Proof. This is just a big diagram chase. We have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f_X"] & Y \arrow[r] \arrow[d, "f_Y"] & Z \arrow[d, "f_Z"] \\ \hat{X} \arrow[r] & \hat{Y} \arrow[r] & \hat{Z} \end{tikzcd}$$

with exact rows. It follows from the snake lemma that we have an exact triangle

$$\mathrm{cofib}(X \to \hat{X}) \to \mathrm{cofib}(Y \to \hat{Y}) \to \mathrm{cofib}(Z \to \hat{Z}).$$

As deirved I-completeness is equivalent to the cofiber of the map $M \to \hat{M}$ vanishing, we conclude since if 2 out of 3 objects in an exact triangle vanish, then so does the third. $\square$

We remark that $\mathcal{D}_{I-comp}(A)$ is moreover stable under truncations. $\mathcal{D}(A)$.

Proposition 7.7. *Fix $I \subseteq A$ a finitely generated ideal. For any $M \in \mathcal{D}(A)$, the following are equivalent:*

(1) each $H^q(M)$ (for $q \in \mathbb{Z}$) is derived I-complete (thought of as a complex in degree 0),

(2) $\mathrm{Hom}_A(A_f, M) = \mathrm{Ext}^1_A(A_f, M) = 0$ for each $f \in I$ (we can even check this on just a generating set),

(3) M is derived I-complete.

In particular, if M is derived I-complete, then each of its truncations (with respect to the usual t-structure on $\mathcal{D}(A)$) are.

Proof. By **??**, M is derived I-complete if and only if

$$R\operatorname{Hom}_A(A_f, M) = 0$$

for every $f \in I$.

There is a classical spectral sequence

$$E_2^{pq} = R^p \operatorname{Hom}_A(A_f, H^q(M)) \implies R^{p+q}\operatorname{Hom}_A(A_f, M).$$

Recall $R^p \operatorname{Hom}_A(A_f, H^q(M)) = \operatorname{Ext}_A^p(A_f, H^q(M))$. Note that A_f classically has a length 2 free resolution

$$A[y] \xrightarrow{y \mapsto fx-1} A[x] \xrightarrow{x \mapsto f^{-1}} A_f,$$

and so $\operatorname{Ext}_A^p(A_f, H^q(M)) = 0$ for all $p > 1$. Thus the spectral sequence degenerates at the E_3-page, and we have a short exact sequence

$$0 \to E_2^{1,p-1} \to E_\infty^p \to E_2^{0,p} \to 0,$$

coming from the fact that, whenever E has a two step filtration, we have a short exact sequence

$$0 \to F^1E = \operatorname{gr}^0(E) \to E \to \operatorname{gr}^1(E) \to 0.$$

Specializing to our case, we get short exact sequences

$$0 \to \operatorname{Ext}_A^1(A_f, H^{p-1}(M)) \to R^p\operatorname{Hom}_A(A_f, M) \to \operatorname{Hom}_A(A_f, H^{p-1}(M)) \to 0.$$

Hence M is derived I-complete iff $R^p \operatorname{Hom}_A(A_f, M)$ always vanishes iff the second condition in our theorem statement holds. The same argument for $H^p(M)$ instead of M shows that condition 1 and condition 2 of our theorem are equivalent, so we conclude. $\square$

We now give a derived form of Nakayama's lemma.

Lemma 7.8. *Let $A \to A'$ be a surjective ring homomorphism, with kernel J being nilpotent of finite order n. Then, for any $M \in \mathcal{D}(A)$, we have $M \otimes_A^{\mathbb{L}} A' = 0$ if and only if $M = 0$.*

Proof. The idea is that $M \otimes_A^{\mathbb{L}} A/J = 0$, and we want to show $M = 0$.

First take N any A-module which has $JN = 0$. Then we can view N as an A/J-module to write

$$\begin{aligned} M \otimes_A^{\mathbb{L}} N &= (M \otimes_A^{\mathbb{L}} A/J) \otimes_{A/J}^{\mathbb{L}} N \\ &= 0 \otimes_{A/J}^{\mathbb{L}} N \\ &= 0. \end{aligned}$$

Next, assume $N \in \operatorname{Mod}_A$ has $J^2N = 0$. We thus have a short exact sequence

$$0 \to JN \to N \to N/J \to 0.$$

Note that both JN and N/J have $J(JN) = J(N/J) = 0$. Thus M tensored with either of them vanishes; so $M \otimes_A^{\mathbb{L}} N$ is an extension of zero by zero, and hence vanishes.

Doing the same argument and inducting, we find that $M \otimes^{\mathbb{L}}_A N = 0$ for any N which has bounded J-torsion; as $J^n = 0$, we find that this argument holds also for $N = A$, and we conclude $M = 0$. □

Theorem 7.9 (Nakayama's lemma)**.** *Let I be a finitely generated ideal of a ring A.*

Take $M \in \mathcal{D}(A)$ a derived I-complete A-module. Then $M = 0$ if and only if $M \otimes^{\mathbb{L}}_A A/I = 0$.

Proof. If $M = 0$, then of course this tensor vanishes.

For the nontrivial direction, assume the tensor product vanishes. Fix a generating set $f_1, ..., f_r$ of I. Then

$$M = \hat{M} = R\varprojlim_n (M \otimes^{\mathbb{L}}_{A[x_1,...,x_r]} A[x_1, ..., x_r]/(x_1^n, ..., x_r^n).$$

Note that, as an A-module (with x_i acting as f_i), $A[x_1, ..., x_r]/(x_1^n, ..., x_r^n)$ is I-power torsion (meaning for every $i \in I, m \in A[x_1, ..., x_r]/(x_1^n, ..., x_r^n)$, there is some $N \gg 0$ with $i^N m = 0$).

We will now $M = 0$ by showing each term in that inverse limit is zero, by proving more generally that if $M \otimes^{\mathbb{L}}_A A/I = 0$ for any M, then $M \otimes^{\mathbb{L}}_A N = 0$ for any $N \in \mathrm{Mod}_A$ which is I-power torsion.

When N is I-power torsion, we may write

$$N = \varinjlim_n N[I^n],$$

and hence (as filtered colimits are exact) write

$$M \otimes^{\mathbb{L}}_A N = \operatorname{colim} M \otimes^{\mathbb{L}}_A N[I^n],$$

so it suffices to prove this for I^n-torsion A-modules.

To leverage our condition about A/I, note that there is a surjection of A-algebras

$$A/I^n \oplus N \to A/I,$$

where we view $A/I^n \oplus N$ as the trivial square zero extension of A/I^n by N (so we give it the obvious additive structure, and give it a multiplication by declaring N to be a square zero ideal). This map has kernel $J = I/I^n \oplus N$, and you can easily check $J^{n+1} = 0$.

Set $A' = A/I^n \oplus N$. Then

$$(M \otimes^{\mathbb{L}}_A A') \otimes^{\mathbb{L}}_{A'} A/I = M \otimes^{\mathbb{L}}_A A/I = 0,$$

and so by Lemma 7.8 we find $M \otimes^{\mathbb{L}}_A A' = 0$. But A' is a direct sum of two A-modules, and so by cocontinuity of the tensor product we deduce that $M \otimes^{\mathbb{L}}_A N = 0$ as well (this was the entire point of introducing the square zero extension by N). We conclude. □