What the **** is a q-number, and why should I care?

cf Habib, Motive Theory Hidden in the Karaji-Pascal Triangle, arXiv:2606.2613v1.

---

§1 Motivation.

let $m,n \in \mathbb{Z}_{\geq 0}$. let K be a field eg. $\mathbb{F}_q$, $\mathbb{C}$.

① let V be a ~~set~~ vs/K, $\overset{\dim}{\#} V = m+n$.

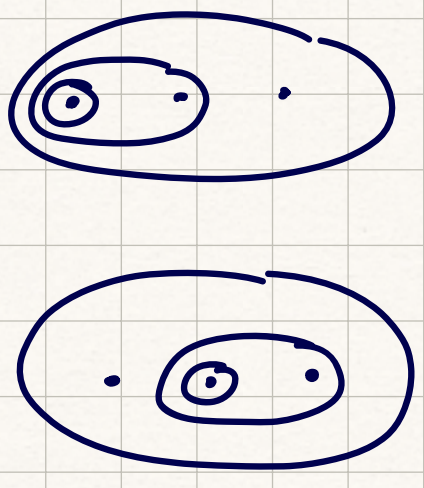
let $Gr^{m,n} = Gr(m,V) := \{ \Lambda \subset V \text{ s.t. } \overset{\dim}{\#}\Lambda = m \}$.

② let W be a set, $\overset{\dim}{\#} W = n$.

A <u>complete flag</u> in W is a seq.

$$\emptyset = W^0 \subset W^1 \subset \cdots \subset W^n = W$$

s.t. $\forall\, i=0,\ldots,n$, $\overset{\dim}{\#} W^i = i$.



let $Fl_n = Fl(W) = \{ \text{Complete flags } \mathcal{W} \text{ in } W \}$.

K	$\bigcirc$	$\mathbb{F}_q$	$\mathbb{C}$
$\lvert Gr^{m,n} \rvert$	$\binom{m+n}{m}$	$\binom{m+n}{m}_q$	$\binom{m+n}{m}_{t^2}$
$\lvert Fl_n \rvert$	$n!$	$[n]_q!$	$[n]!_{t^2}$

What is $\#Gr^{m,n}(\mathbb{F}_q)$?

Example: $\#Gr^{1,n}(\mathbb{F}_q) = \#\{\ell \subset \mathbb{F}_q^{n+1}\} = \frac{q^{n+1}-1}{q-1} = \underbrace{1+\cdots+q^n}_{[n+1]_q}$.

Alt pf: $GL_{m+n}(\mathbb{F}_q) \curvearrowright Gr^{m,n}(\mathbb{F}_q)$ & stabilizer is

$$\begin{array}{c|cc} & m & n \\ \hline m & * & * \\ \hline n & 0 & * \end{array}$$

$$\therefore \#Gr^{m,n}(\mathbb{F}_q) = \frac{\#GL_{m+n}(\mathbb{F}_q)}{\#GL_m(\mathbb{F}_q) \cdot \#GL_n(\mathbb{F}_q) \cdot q^{mn}}$$

$$= \binom{m+n}{m}_q \; ?$$

Recall: $[n]_q := \frac{q^n - 1}{q-1} = 1 + \cdots + q^{n-1} \in \mathbb{Z}[q]$.

$[n]_q! := \prod_{i=1}^{n} [i]_q$ and $\binom{m+n}{m}_q := \frac{[m+n]_q!}{[m]_q!\,[n]_q!}$

Ex: $\binom{4}{2}_q = 1 + q + 2q^2 + q^3 + q^4$.

Rmks. ① $\in \mathbb{Z}[q]$. Agnostic.

② What is q? — Prime power, formal variable, ...

③ $\lim_{q \to 1} [n]_q = n$ etc.

Over $\mathbb{C}$: Compact $\mathbb{C}$-mfld $\therefore$ Not clear what dim? No.

Ans. Topological Euler characteristic: $V - E + F - \cdots$

Can even upgrade to Poincaré polynomial = gen. series of Betti #s.

Goal: (i) Conceptual explanation of why.

(ii) What goes here & why you should care?

---

§2 q-Pascal's Identity

not correct, but we'll see correction

$$\binom{m+n}{m}_q = \binom{m+(n-1)}{m}_q + q^n \binom{(m-1)+n}{m-1}_q .$$

3 interpretations.

① Pascal Δ

②

Pick any $H \subset V^{m+n}$ s.t. $\dim H = m+n-1$. Then

$Gr(m,V) = Z \sqcup U$ where $Z = \{\Lambda \subset V : \Lambda \subset H\}$

$U = \{\Lambda \subset V : \Delta \not\subset H\}$.

Then $Z \simeq Gr(m, H) \simeq Gr^{m,n-1}$ and

$U \longrightarrow Gr(m-1, H) \simeq Gr^{m-1,n}$.

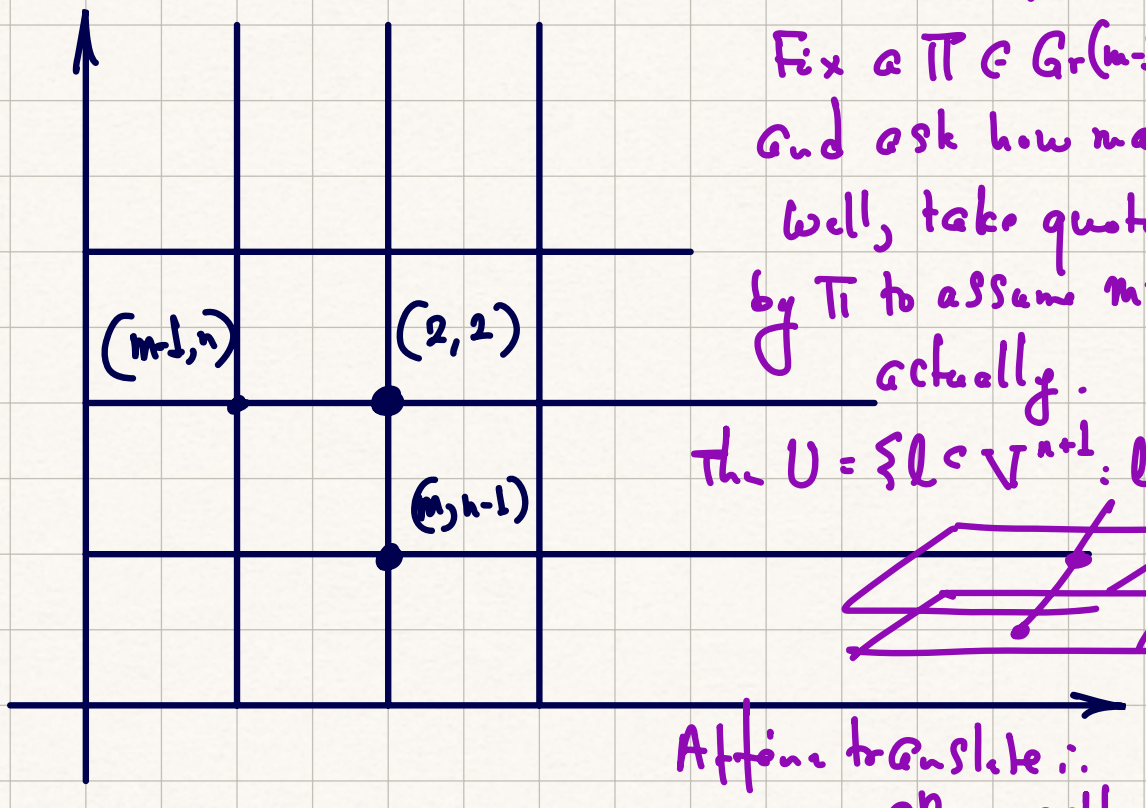
$\Lambda \longmapsto \Lambda \cap H$.

How many?

Fix a $\Pi \in Gr(m-1, H)$ and ask how many? Well, take quotient by Π to assume $m=1$ actually.

Then $U = \{\ell \subset V^{n+1} : \ell \not\subset H\}$

③ Grid.

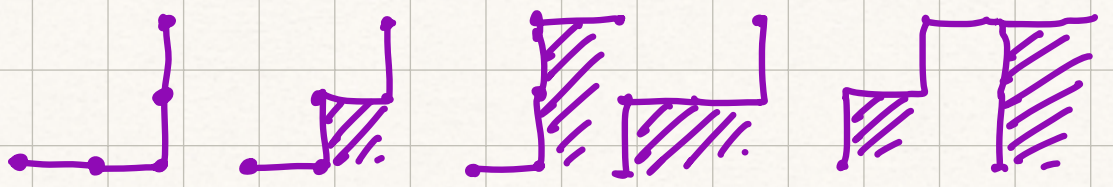


Affine translate: $\mathbb{P}^n$ exactly

$$\binom{m+n}{m}_q := \sum_{P:(0,0)\to(m,n)} q^{\mathrm{Area}(P)}$$

Eg. $m=1$ gives $n+1$ exactly. $1+q+\cdots+q^n$.

$m=n=2$ gives 6. $1+q+2q^2+q^3+q^4$.



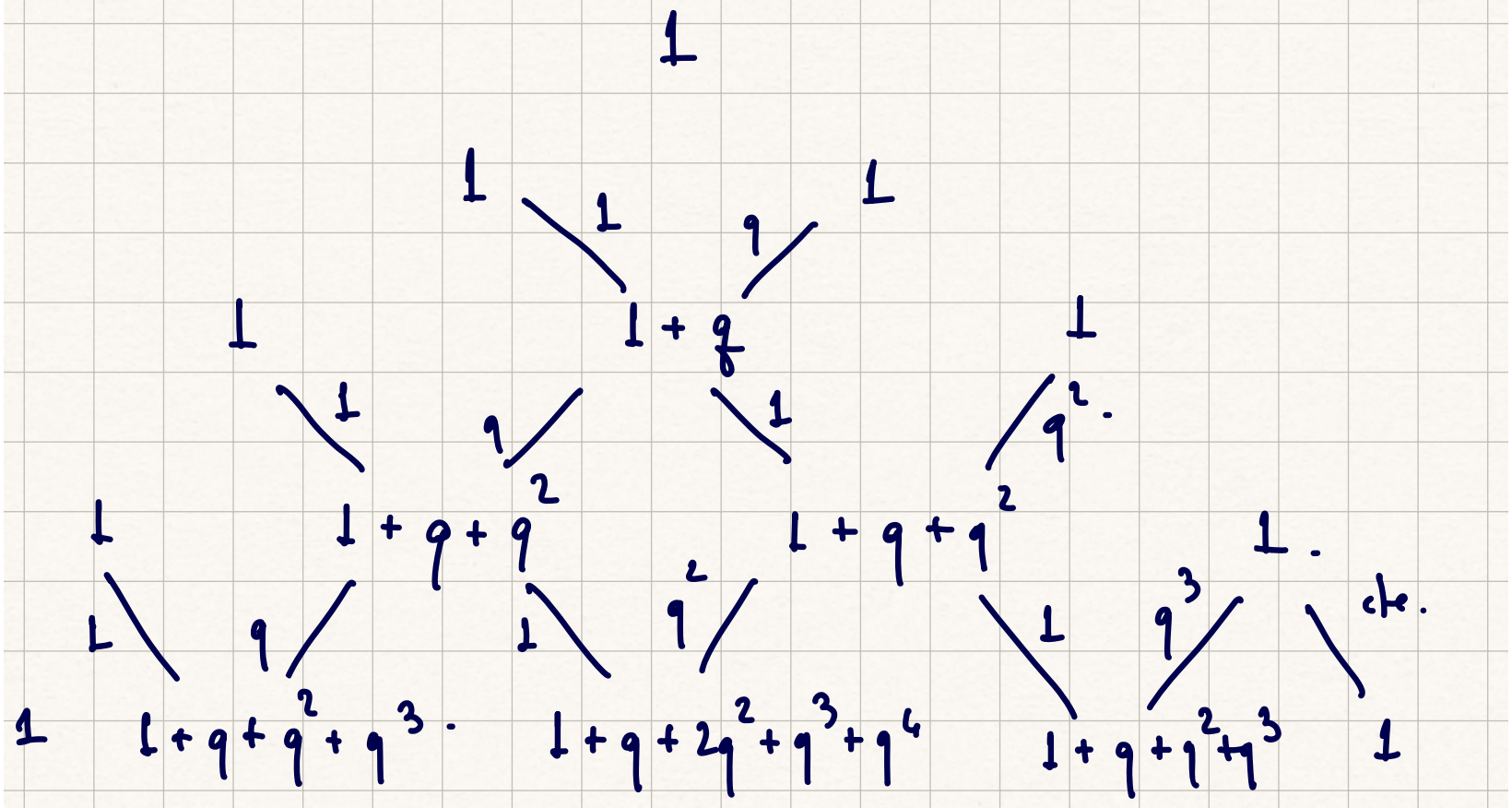
In this, just look at penultimate step.

Again just look at penultimate step. Then q-Pascal Δ: clearly determined by this.

Thm $\#Gr^{m,n}(\mathbb{F}_q) = \sum_{P:(0,0)\to(m,n)} q^{\mathrm{Area}(P)} = \dfrac{[m+n]_q!}{[m]_q![n]_q!}$

Pf. Show outer 2 satisfy same recurrence. For ! elem.
For $Gr^{m,n}$, using orbit-stabilizer, but better pf by recurrence!





(§3) $\mathbb{C}$: Some decomposition, $Z \simeq Gr^{m-1,n}$

$U \simeq \mathbb{A}^n$-bundle over $Gr^{m,n-1}$.

$\therefore \chi(Gr^{m,n}) = \chi(Z) + \chi(U)$ ——— little thought

Poincaré poly.

Really: Affine paving gives motivic decomp. & these are all manifestations

§4 What goes here? Of course, $\mathbb{F}_1$!

In schemes, final object = $\operatorname{Spec}\mathbb{Z}$, one-dim'l — Highly non desirable
Top spaces, ——"—— is $*$.

Hope in Sch^+, final object = $\operatorname{Spec}\mathbb{F}_1$.

$$\operatorname{Spec}\mathbb{Z} \overset{?}{\simeq} \mathbb{A}^1_{\mathbb{F}_1} \qquad \mathbb{A}^1_{\mathbb{F}_q}$$
$$\updownarrow \qquad\qquad \updownarrow$$
$$\mathbb{F}_1\,?? \qquad\qquad \mathbb{F}_q$$

This is what I meant by my cryptic comment.

Why? ① Well, assuming such analogy, formulate analogs of RH on RHS, which are true!
② Analogies start to make sense b/w Knots in S^3 and number rings etc. (Feng).
③ Matroid theory... shadows of "$\mathbb{F}_1$".

∴ Searching for the right content. Band schemes, tropical geometry, who knows? ▨