Observations on Vakil's
The Rising Sea: Foundations of Algebraic Geometry

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Important Disclaimer: The following list contains my observations on the content of The Rising Sea. This list is meant to complement and expand the list of errata found the website, and not replace it. It is maintained by me (Dhruv) and has not been verified or ratified by Vakil. Proceed at your own risk.

The following errata and addenda are generally labelled via pairs (S, L), where S stands for minimal subsection (e.g. exercise, example, etc.), and L for the line number (or description of location). Sometimes L is omitted if it does not add to the clarity of expressing the location. If the reader encounters any further errors in the book, or errors in this errata list, they are highly encouraged to (electronically) write to me at either this address or this one.

Contents

1 Substantial Corrections or Observations

1. (4.4.1) "It is not enough to show that (x, y) is not a principal ideal": I am not sure I understand this; I think it is enough to do so. Here's why: if $U = D(f)$, then $\mathbb{V}(f) = \{0\}$. Then $\sqrt{(f)} = \mathbb{IV}(f) = (x, y)$, but since $k[x, y]$ is a UFD, $\sqrt{(f)}$ is principal and generated by $\mathrm{rad}(f)$.
2. (7.5.4) I think we need more assumptions on X and Y to define the composition of rational functions. For instance, we need X to be integral (or at least that each irreducible component of X dominates one of Y). Otherwise, the map π could contract a component of X to land in the indeterminacy locus of ρ, and then the composition would not be defined on that component at all.
3. (7.6.K) I think it is worth emphasizing that this works only if we assume that the necessary products exist in the relevant category–this is not automatic. See, for instance, [1]. (We're probably implicitly assuming this given the notation, but I think it would be helpful to clarify this–I was certainly confused by this.)
4. (8.4.4)
   (a) (8.4.7) I think $\mathrm{FL}(\delta Z) \subset \mathrm{FL}(\overline{Z})$ allows us to write simply $Y' = Y \setminus \mathrm{FL}(\overline{Z})$. I think that 8.4.7 could be made less verbose and clearer by simply writing down

$$Y \setminus \pi(Z) = \mathrm{FL}(\delta Z) \amalg (Y' \setminus \pi(Z')).$$

(b) "Z' is quasifinite over Y'" We have only defined what it means for a morphism to be quasifinite, and we haven't learned how to give Z' a scheme structure yet. I guess in retrospect it comes from writing $Z = \mathbb{V}(f_1(t), \ldots, f_r(t))$ at the top of the page, but I think that it is worth emphasizing that this is *structure* or at least *additional data* we are specifying at the beginning of 8.4.5 and depends on the choice of the f_i's. As currently written, it seems like Z' (and by extension Z) came with some "natural" scheme structure, which is not true, given we haven't gotten to 9.4.9 yet.

5. (10.5.P) The last bit of part (a) ("as well as in $F(x_1, \ldots, x_{d-1}, T)[x_d]$") is incorrect as stated, since the variable x_d may not appear altogether. I think we need to include "after reordering if needed" in the statement. Alternatively, the statement "as well as in $F(x_1, \ldots, x_{d-1}, T)[x_d]$" can be moved to after we have relabelled as in part (b) so that $j = d$.
6. (12.4.C) I think this exercise is harder than intended. The result is easier if we ask X and Y to be of finite type over a field (see the question in [2]); this is all we need to continue. The only proofs I know of the result as stated involve either generic flatness ([2]) or Zariski's Main Theorem [3, 05F6].
7. (12.2.K) The definition of local dimension 12.2.K is not literally the same as the one ([3, 04MT]) in the Stacks Project (or EGA IV_1.0.14.1.2), but I think these are in fact equal, at least in the case that we're considering: finite-dimensional Noetherian X. (It differs from the one in [4, III.9.5].)
8. (13.2.7(a)) This is false as stated right now, only for the silly reason that the definition of smoothness over a field (13.2.4) is incorrect in the text (as is acknowledged on p. 239). For instance, the disjoint union of an $\mathbb{A}^1_k$ and an $\mathbb{A}^0_k$ is regular but not smooth according to 13.2.4. Similarly, in 13.2.J, we should probably remove the "of pure dimension d" from the statement, since it is not part of the claim being made. (Also, the hint can be improved by invoking 13.2.E.)
9. (14.1.K(a)) "a fractional ideal is the same as an invertible sheaf *with a trivialization at the generic point*" $\to$ "a fractional ideal is the same as an invertible sheaf with a trivialization at the generic point, up to scaling by an element of $\mathscr{O}_K^\times$". (The statement is not true otherwise, at least if we understand trivialization to mean a K-linear isomorphism of $\mathscr{L}_\eta$ with K.)
10. (14.2.1) As far as I can tell, we have only defined rational functions/maps on locally Noetherian or reduced schemes, so we should also ask X to be locally Noetherian or reduced when making this definition.
11. (Right before 14.5.6) The last line should read "the pullback π^* exists if Y is an open subset of an affine scheme *and the sheaf is pulled back from a quasicoherent sheaf on this affine scheme*".
12. (15.2.N,O) These exercises are incorrect as stated (certainly as intended). The condition that for all $y \in Y$, there are affine opens $U \subset X$ and $V \subset Y$ with $y \in V$ and $\alpha^{-1}(U) = \beta^{-1}(V)$–which is automatic when β is a closed immersion–is unfortunately not automatic in this case; if it were, we could contract affine closed subschemes of non-affine schemes and still remain in schemes, which is not true in general. Here's an explicit counterexample: let $E \subset \mathbb{P}^2_{\mathbb{C}}$ be an elliptic curve in Weierstrass form (say), and $e = [0, 1, 0]$ the point at infinity. Pick a non-torsion point $p \in E(\mathbb{C})$. Take $Z = E \setminus \{p\}, Y = \operatorname{Spec} \mathbb{C}$ and $X = \mathbb{P}^2 \setminus \{p\}$. Then β is affine because Z is (19.2.C), but the resulting pushout in locally ringed spaces is not a scheme in such a way that β' is affine. Here's why: if it is, take an affine neighborhood of the image of E and consider its preimage in $\mathbb{P}^2$. This is an affine open containing $E \setminus \{p\}$. Every affine open in $\mathbb{P}^2$ is distinguished, so write this as $D(f)$ for some f, say of degree $d > 0$. Then if $C = \mathbb{V}(f)$, then $C \cdot E = 3d(p)$, which tells us that p is $3d$-torsion, which it is not. I think we need to be careful about the kinds of β we allow: we are OK if β is integral and for all $y \in Y$ there is an affine open $U \subset X$ such that $\beta^{-1}(y) \subset \alpha^{-1}(U)$ (e.g. if β is a closed immersion), or if α is a thickening (c.f. [3, 0ECH, 07RS, 0E35] and ther Ferrand paper linked there, which gives sufficient conditions.) By the way, this gluing construction gives a much simpler example for 19.11.H: glue $\mathbb{A}^2_k$ and $\mathbb{P}^2_k \setminus \{p\}$ along $\mathbb{A}^1_k$. (If we want, this is even quasiprojective: it embeds as an open subset of the union of two planes in $\mathbb{P}^3_k$.) This has global sections $k + xk[x, y]$ which is not Noetherian.
13. (15.3.B) I only know how to define the "domain of definition" when X is reduced, using 11.3.3. How do we define this otherwise?
14. I found chapter 17 quite difficult to read. Here are three specific comments:

    (i) (Exercise 17.3.A) Do we really intend the reader to discover all closed subschemes of relative Proj on their own? (That's the only way I see how to do that problem; that's how EGA does it.)

    (ii) (17.3.6) As far as I can tell, the last statement "If $\mathscr{L}$ is π-very ample and $\mathscr{M}$ π-bpf, then

$\mathscr{L} \otimes \mathscr{M}$ is π-very ample" does not immediately follow from 16.2.C, since being relatively very ample is not local. (I am not convinced it is true, given our choice of definitions. I think we should require Y to be locally Noetherian and invoke 17.3.H.)

(iii) (17.3.G) In the proof, (a') $\Rightarrow$ (a) $\Rightarrow$ (c) $\Rightarrow$ (b) is clear, but (b) $\Rightarrow$ others needs tools not introduced. The only way I see of doing it is when Y is qcqs, we can approximate a qcoh sheaf by finite-type qcoh sheaves (as done in [5, Chapter 13], which cleanly does this when Y is qcqs). What is the intended solution for (b) $\Rightarrow$ others in 17.3.7/G?

15. (18.4.C) "which is regular at all singular points of C" $\to$ "which is regular and nonvanishing at all associated and nonregular points of C", i.e., let S be the set of associated and nonregular points of C; then there is an open $U \subset C$ containing S and $s \in H^0(U, L)$ such that $S \subset D(f)$.
16. (18.6.K) How do we show easily at this point that the "no embedded points" hypothesis is stable under passing to the algebraic closure? The only way I know of doing this is basically to invoke 24.1.K or EGA IV_2.3.3.2.
17. (18.6.L) "although we are free to remove this hypothesis if we wish." The result as stated is false in general over non-algebraically closed fields, if we only require C to be integral and not geometrically irreducible. For instance, we can take two Galois-conjugate skew lines in $\mathbb{P}^3_{\mathbb{C}}$; this descends to an integral degree 2 closed subscheme of $\mathbb{P}^3_{\mathbb{R}}$ of dimension 1 for which the result is false. Maybe it's worth explicitly stating that the right hypotheses are "geometrically irreducible with no embedded points".
18. (18.6.5) "Do we see why?" I do not see this at all. This seems to me to be somewhat nontrivial ([6, Theorem 1.5.8]). Is there an easier way to see that a general hyperplane section of a complex variety is again a variety (i.e. reduced)?
19. (18.9.4) At the very end (after 18.9.G), it is not clear to me that $\mu^{-1}(U_i) = A_i$. I don't think that is actually true in general, or that we need it. We only need that the A_i cover X, which follows from the containment $\mu^{-1}(U_i) \subset A_i$; this is what EGA II.5.6 does. To show $\mu^{-1}(U_i) \subset A_i$, Grothendieck uses an argument involving the schematic closure which seems nontrivial (or at least worth mentioning); it uses that $\overline{U_i}$ is separated and that $\mu^{-1}(U_i)$ is the schematic closure of U in $U_i \times \prod \overline{U_i}$, so it suffices to check the result after precomposing with U, which is OK.
20. (Introduction to Chapter 19) At the top, the second line should say "projective, geometrically integral, and smooth (or equivalently geometrically regular) curves over a field k". In general, I am not very happy with the usage of the term "(geometrically) regular" in this section (19.2.1, 19.2.F, 19.3.B, etc.), (a) because we haven't defined or discussed this notion, (b) because it is not obvious (to me) that this implies regularity–for instance, if the definition is that geometric fibers are regular–(c) because we have already talked about smoothness and (d) because smoothness, and not regularity, is what we really need. I think it is best to globally replace the word "regular" in the entire chapter with "smooth".
21. (19.5.C) The term "hyperelliptic involution" has not been defined up to this point of time in the book. (Is part of the exercise also finding the definition of such a thing?) We also probably want to use a different letter for the automorphism of C.
22. (19.7.A) "Feel free to remove this hypothesis" I don't think it is very easy to remove this hypothesis. For starters, we have not even defined what it means for a curve over an arbitrary field to be hyperelliptic. Next, if we take the definition to be "there is a $2:1$ separable cover of a genus 0 curve", then it is not obvious to me that C not hyperelliptic implies $C_{\overline{k}}$ is not hyperelliptic: I think this follows from Galois descent on the Brill-Noether locus to show that the unique g^1_2 is preserved under automorphisms. (I have no idea what to do over imperfect fields, either.) How can we remove this hypothesis?
23. (19.8) We should probably assume that our field has characteristic $\neq 2$ to be able to talk about the classification of quadric surfaces. Again, it might be clearer to use the word "smooth" and not "regular" here, or to assume that k is algebraically closed. Is there a way to avoid these hypotheses that I am failing to see?
24. (19.8.C) "Show that it lies on a three-dimensional vector space of quadric surfaces." This could mean two things: it could mean for us to show that this dimension is ≥ 3 or $= 3$. That it is ≥ 3 is very easy, but $= 3$ is much harder (I think). The only argument I know is the one found in Eisenbud-Harris's book on curves ([7, Theorem 9.10]), which I learnt when I took the corresponding class on curves with Harris. That argument I remember being somewhat involved (using Lasker's unmixedness theorem, etc.). Is there an easier argument?
25. (20.2.R) I think the "$n \geq 0$" should be "$n > 0$". The result is also incorrect as stated when we take "curve" to mean what it has so far, namely effective Cartier divisor. For example, $2E + 3F$

on $\mathbb{F}_4$ has self-intersection -4. I think the result should say E is the unique irreducible curve on $\mathbb{F}_n$ with self-intersection $-n$ (even unique integral curve of negative self-intersection, which one could potentially need in solving 28.6.K). Finally, when showing that there are no reduced curves on $\mathbb{F}_n$ of smaller self-intersection, I think we need properties of Cohen-Macaulay schemes to say that an effective Cartier divisor $D = \sum a_i C_i$ is reduced iff each $a_i = 1$, i.e., there couldn't be any embedded points obstructing reducedness.

26. (21.2.16) I don't think the implication (a) $\Rightarrow$ (b) follows automatically, given our (nonstandard) definition of a regular closed embedding in the non-locally-Noetherian setting (9.5.7 vs. [8, 19.19] or [3, 0638]), although the problem does not arise when the ideal sheaf and hence the conormal sheaf is of finite type ([8, 19.20]). I am concerned about cases in which the closed subscheme is cut out by a quasicoherent ideal sheaf that is not of finite type, and it is regular in the sense of 9.5.7 but not [8, 19.19]; in this case, 21.2.16 (along with 21.2.26(b)) only tells us that the stalks of the conormal sheaf are locally free, but that might not be enough in the non-locally-Noetherian setting to invoke 14.3.E to conclude that the conormal bundle is locally free. I would suggest changing 9.5.7 to the definition in [8, 19.19] or [3, 0638].
27. (Proof of 21.4.2, Line -5) We don't know that $\mathscr{O}_\eta \otimes \Omega_{X/k}$ is an n-dimensional vector space over $K(X)$ a priori, since we don't know that $\Omega_{X/k}$ is locally free: we haven't assumed X is smooth and haven't proved generic smoothness yet. But the exactness of the sequence along with the fact that $\mathscr{O}_\eta \otimes \Omega_{X/Y} = 0$ forces $\mathscr{O}_\eta \otimes \Omega_{X/k}$ to have dimension n as well (since we know by that it is $\geq n$ by 13.1.M), and forces the map to be injective as in the proof.
28. (21.4.A) We haven't defined what it means for the map π to be generically separable (21.4.1 only applies to integral X and Y). It is perhaps best to ask X and Y to be irreducible, or to modify 21.4.1 to cover the reducible case by asking that every component of X dominates a component of Y and the extension of function fields is separable on each component of X dominating a component of Y.
29. (After 21.4.A) "More precisely, suppose π is an isomorphism over an open subset of Y, from an open subset U of X whose complement has codimension greater than 1. Then Y cannot be smooth." This is not correct as written: consider the identity map $\mathbb{A}^2 \to \mathbb{A}^2$, take $U = V = \mathbb{A}^2 \setminus \{0\}$. Then $\operatorname{codim}_X(X \setminus U) = 2$, we have an isomorphism $U \to V$, the map contracts $\mathbb{A}^2 \setminus U$ to a point, but $\mathbb{A}^2$ is smooth. We probably need to ask for $X \setminus U$ to have a positive-dimensional component of codimension ≥ 2 that gets contracted to a point.
30. (21.6.4) I just cannot make sense of this proof. Here are concrete problems:
    (i) "The minimal polynomial may be interpreted as": I'm not sure what this means. Take $B \to A$ to be $k \to k[x,y]/(xy-1)$ so $n = 1, N = 2, x_1 = x, x_2 = y$. Then the minimal polynomial of y is $t - (1/x)$, which is not an element of $k[x,t]$. In a similar vein, I'm not sure I understand what "a minimal polynomial" is. If it is taken to mean an arbitrary multiple of *the* minimal polynomial, then the proof doesn't work, since it might introduce extraneous components. We need to "minimally clear denominators", but it's not clear to me what this means beyond the first step, when we need not be in UFDs.
    (ii) "because $A' \otimes_B K(B) \cong K(A)$" This is incorrect. Take $B \to A$ to be $k \to k[x]$, so $n = N = 1$. Then $A' = k[x]$ and so the LHS is $k[x]$ while the RHS is $k(x)$. A similar example in a higher number of variables (and hence higher n, N) can be constructed. Even if the LHS is a domain (which needs some minimal clearing of denominators), we can only say the RHS is the fraction field of the LHS.
    (iii) "This is the open subset we seek" I'm not fully convinced this is sufficient to conclude, since we need a specific map to be smooth. We probably want to say something like $A' \twoheadrightarrow A$, making $\operatorname{Spec} A \hookrightarrow \operatorname{Spec} A'$ an irreducible component.

    I've attempted a proof at the level of this exposition here: [9, Lemma 4.3.13].
31. (21.6.7) "We can replace X by an irreducible component of X_r": In general, there is no natural subscheme structure on an irreducible component of a constructible subset of a scheme; consider for example the standard example of removing a line and adding back in a point in $\mathbb{A}^2$, which is irreducible. Perhaps we should say: X_r is a finite disjoint union of locally closed subsets, pick one, and replace X by an irreducible component of that. Also, right before this we have "$\pi(X_r)$ is constructible, so we can make sense of $\dim \pi(X_r)$". I'm not sure what this means: can't we make sense of the dimension of any topological space, and what constructibility really gives us is that this notion is "reasonable", i.e., agrees with the dimension of the closure?
32. (21.6.8) There is a forward reference to 24.8.G(b) here. Also, when we say "so remove $\overline{\pi(X_{n-1})}$,"

we better assume at some stage early on in the proof that Y is irreducible of dimension n, else we might remove smaller dimensional components, so the open set produced will not be dense.

33. (21.6.10) The statement is incorrect as written, but only if I am being very pedantic. (Take $X = Z = *$, G to be any group, and Y to be something with 2 components of different dimensions.) But that means that it is not easy to prove as written. Kleiman and Hartshorne include integrality for Y and Z. I think the right hypotheses—stable under field extension—are that Y and Z are nonempty and pure-dimensional. I also think it is better to change "nonempty open" to "dense open" in (a) and (b); this is very helpful when dealing with 21.6.H when k is not algebraically closed.
34. (21.6.H) I don't think the reader has enough tools to do this proof at this stage, and I think it is best done after the language of flatness (even generic flatness, which we don't prove). Here's the issue: We can show that $G \times Y \to X$ is surjective with fibers pure of dimension $g + y - x$, so $(G \times Y) \times_X Z \to Z$ is surjective with fibers pure of dimension $g + y - x$, but this doesn't imply that $(G \times Y) \times_X Z$ is pure of dimension $g + y + z - x$ unless Y and Z are pure dimensional and map is flat. A priori, there may be components of smaller dimension, and these can dominate G in general. I think the way around is to use generic flatness (before passing to the algebraic closure, because we are not assuming geometrically reduced). As an alternative, I can think of a proof without using generic flatness when X is smooth. I have included both these explanations in [9, Hint to Exercise 4.14]. Perhaps there is a different way around this.
35. (24.2.K) The statement in parentheses ("if we wish, ...") is incorrect as written. The cohomology of the complex $C^\bullet \otimes_A F^\bullet$ is always zero, since we are assuming that $F^\bullet$ is an exact complex of flat modules.
36. (24.3.6) The (a)-(c) are equivalent and imply (d) when M is finitely generated, but I am not convinced (d) implies the others when M is not finitely presented. At least, as far as I can tell, Matsumura's proof crucially uses the relational criterion for flatness for (c) $\Rightarrow$ (a), and ultimately it needs some more Tors to vanish. It is not clear (to me) how (d) $\Rightarrow$ (a) in the finitely generated, non-Noetherian case, even after reading Matsumura.
37. (24.7.D) We should probably ask for Y to be locally Noetherian to apply the previous results. Also, I do not think we need π to be flat.
38. (Proof of 24.8.7) "Finally let $R = P_J$". I'm not sure what this even means: J is a polynomial in the variables $x_1, \dots, x_{n+r}$. The following Cartesian diagram also does not quite make sense to me. I think we should have $\operatorname{Spec} P$ on the bottom right, $\operatorname{Spec} P[x_1, \dots, x_{n+r}]/(g_1, \dots, g_r)[1/J]$ on the top right, and $\operatorname{Spec} B[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)/[1/\text{Jac.det.}]$ on the top left (using the observation that inverting elements doesn't affect smoothness).
39. (25.2.9) I don't see at all how we are invoking Lemma 25.2.8. As far as I can tell, the argument works when q is a closed point, but not any point, and we have not explicitly reduced to that case.
40. (25.2.C(c)) "we will have to show that $\mathbb{Z}[x_1, \dots, x_N]/I$ can be taken to be reduced" I don't see this at all. Everything follows from the existence of the Mumford complex, so we don't need this; and if $g : B \to B'$ is a map, $f : B' \to \mathbb{Z}$ a function, and fg is locally constant, this does not imply f is locally constant in general.
41. (26.1.B and E): I think that either we should fix a convention on the depth of the zero module, or insist in all questions of depth that the involved modules are nonzero.
42. (27.3.H) The exercise is incorrect as stated: we also need to exclude the case of exactly eight points, all lying on a singular cubic, with one at the singularity. Even with this correction, I think this exercise is too hard. The problem is that if we use the Nakai-Moishezon criterion, the problem reduces to showing that if $C \subset \mathbb{P}^2_k$ is an integral curve of degree d passing through points $p_1, \dots, p_n$ for $n \leq 8$ with multiplicities m_i, then $3d > \sum m_i$. The only proof of this result I know uses that $\sum m_i(m_i - 1)/2 \leq (d-1)(d-2)/2$ (and some heavy numerology; see [9, Hint to Exercise 7.2]); but this result has not been covered in this book. Is there a different way around this problem?
43. (28.2.1) ("How does this compare to the definition of node in 13.3.2?") I think that this is a much harder question than intended, because 13.3.2 allows things like $k[x, y]/(xy, y^3)$. The implication 28.2.1 $\Rightarrow$ 13.3.2 is clear, but the only proof of 13.3.2 $\Rightarrow$ 28.2.1 I could come up with (which crucially uses reducedness) uses things like excellence and quasiunmixedness, along with properties of primary decompositions in UFDs (it can be found at [10, Theorem 9.5.1]). Is there an easier proof I am missing?
44. (28.2.H) Are we still working over an algebraically closed field? I don't know how to pass to the algebraic closure otherwise, since that could (while preserving the arithmetic genus) make things very reducible or non-reduced. I think it might be best to make a blanket statement at the

beginning of §28.2 saying we're only going to work over an algebraically closed field.

45. (28.7.2.2 and 28.7.2.3): Should β_n should be b_n? (The notation β_n and β has not been introduced; unless it is taken to mean the map induced by the b_n and the limit thereof.) Also, and more importantly, I think the second equality in 28.7.4.2 crucially uses Noetherianity and needs justification that has not been provided in the book (but can be found in the proof of [4, Prop. III.2.9]).

2 Minor Typos/Corrections

1. (12.3.C) "Let $\{p_1, \ldots, p_m\} \in \mathbb{P}^n_k$" → "Let $\{p_1, ..., p_m\} \subset \mathbb{P}^n_k$"
2. (12.3.L(b)) "any point $p \in X$" → "any closed point $p \in X$". (Also, I think "Chevalley's Theorem 8.4.2" should become "Elimination Theory 8.4.10". I don't see how Chevalley helps at all.)
3. (13.1.4) There is an extra parenthesis at the end.
4. (14.1.0.1) The horizontal arrows should go the other direction to be consistent with the convention for ϕ_i mentioned on the next page.
5. (14.3.C) The first sentence ("Suppose $\mathscr{F}$ is a sheaf of abelian groups.") should be removed.
6. (14.5.1(a)) "when π can be interpreted as some type of "inclusion"" is redundant and should be removed.
7. (14.5.9, Line -1) "Note also that if μ is an exact functor" → "Note also that if μ^* is an exact functor"
8. (15.2.L) "such that $Z \cap \operatorname{Spec} A_1 = Z \cap \operatorname{Spec} A_2$" → "such that $x \in Z \cap \operatorname{Spec} A_1 = Z \cap \operatorname{Spec} A_2$".
9. (15.4.6) "Assume now that X is normal..." should be removed, given the convention at the top of 15.4. (Alternatively, → "Recall that X is normal...").
10. (15.5.A) "issomorphism" → "isomorphism"
11. (15.5.F) Delete the word "earlier" from before "in Exercise 7.6.S(a)".
12. (15.5.R) "$A = k[a, b, x_1, \ldots, x_n]/(ab - x_1^2 - \cdots - x_\ell^2)$" → "$A = k[a, b, x_1, ..., x_\ell]/(ab - x_1^2 - \cdots - x_\ell^2)$"
13. (15.5.Q) We should require x to be nonzero (or equivalently a prime element); this does not follow from (x) being prime.
14. (15.6.B) "When X is normal" → "When X is Noetherian and normal" We haven't defined what $\mathscr{O}(D)$ means otherwise.
15. (16.2.L) "and $q_1, \ldots, q_m$ are closed points on C" → "and $q_1, \ldots, q_m$ are points on C". Also, "where the p_j are regular points" → "where the p_j are regular closed points". (I don't feel strongly about inclusion or exclusion of the word "closed" in front of points generally, but this instance feels like an exception.)
16. (17.1.7) "Exercise 14.3.A(b)" → "Exercise 14.3.A(a)"
17. (Proof of 18.1.6) "is closed (in $\mathbb{P}^n_A$," → "is closed (in $\mathbb{P}^n_A$),"
18. (18.4.E) We should require that $d > 0$.
19. (Proof of 18.5.6, Line −8) "(Theorem 18.1.2(b))" → "(Theorem 18.1.2(c))"
20. (18.8.A) "each quotient $\mathscr{I}^n\mathscr{F}/\mathscr{I}^{n-1}\mathscr{F}$" → "each quotient $\mathscr{I}^n\mathscr{F}/\mathscr{I}^{n+1}\mathscr{F}$"
21. (19.8.3) "Start with the complete intersection of $G(2, 5)$" → "Start with $G(2, 5)$" (We're taking the complete intersection in the next sentence. The first sentence doesn't make sense as written currently.)
22. (19.9.D) "Show that the flexes of the cubic are the 3-torsion points in the group E." → "Show that the flexes of the cubic are the 3-torsion points in the group $E(k)$."
23. (19.10.3) "Since the line $\overline{PQ}$ contains $p\,q$ and r" → "Since the line $\overline{PQ}$ contains P, Q, and R"
24. (After 20.1.6) "two line bundles are linearly equivalent if their degrees are the same on any curve." → "two line bundles are numerically equivalent if their degrees are the same on any curve."
25. (20.2.9) "$\pi^*\mathscr{O}_X(b)^2 = 0$" → "$\pi^*\mathscr{O}_B(b)^2 = 0$".
26. (20.4.5)

Step 1. "The base case is dimension 1, which was done in Exercise 19.2.H" → "The base case is dimension 1, which was done in Exercise 19.2.J", and "closed subvariety" → "proper closed subvariety/scheme".

Step 2. "By Exercise 16.2.C, $\mathscr{L}$ is the difference of two very ample line bundles" → "By Exercise 16.2.F, $\mathscr{L}$ is the difference of two very ample line bundles"

27. (20.4.B(b)) "ample for all $\epsilon \in \mathbb{Q}^{\geq 0}$" → "ample for all $\epsilon \in \mathbb{Q}^{>0}$".
28. (Proof of 20.4.C(a)) "is the limit of ample classes by Exercise 20.4.B(a)" → "is the limit of ample classes by Exercise 20.4.B(b)"

29. (Proof of 21.6.4) "For $i = 0, \ldots, n$" $\to$ "For $i = 0, \ldots, N$"
30. (21.6.7) "$\pi^*\Omega_Y|_{\pi(p)}$" $\to$ "$\pi^*\Omega_Y|_p$"
31. (21.6.I) "let Y be the "universal hyperplane" over $\mathbb{P}V^\vee$ (the incidence variety $I \subset \mathbb{P}V \times \mathbb{P}V^\vee$ of Definition 13.4.I)" $\to$ "let $Y \to X$ be the inclusion of a hyperplane"
32. (Right before 22.3.4) "$\operatorname{Proj} B[X, Y]$" $\to$ "$\operatorname{Proj} B[X, Y]/(yX - xY)$"
33. (24.1.5) "Then there is an exact sequence of $B_\mathfrak{p}$-modules" $\to$ "Then there is an exact sequence of $B_\mathfrak{q}$-modules"
34. (24.1.O(b)) "quasicoherent sheaf of algebras" $\to$ "quasicoherent ideal sheaf"
35. (Right before 24.2.J) "the augmented Čech complex of sheaves for $X = \{U_i\}$, denoted $C^\bullet_{X,\mathrm{aug}}(\mathscr{F})$" $\to$ "the augmented Čech complex of sheaves of $\mathfrak{U} = \{U_i\}$, denoted $C^\bullet_{X,\mathrm{aug}}(\mathfrak{U})$"
36. (24.4.J) "finite-generated" $\to$ "finitely-generated"
37. (24.6.4) "Then T is A-linear" $\to$ "Then T is B-linear"
38. (24.6.9) "We can compute $\operatorname{Tor}_1^B(B/tM, B/\mathfrak{n})$" $\to$ "We can compute $\operatorname{Tor}_1^B(M/tM, B/\mathfrak{n})$" and "$M/tN$ is flat over B" $\to$ "M/tM is flat over B"
39. (24.7.A) "Suppose $X \to Y$ is a projective flat morphism" $\to$ "Suppose $X \to Y$ is a projective flat morphism with Y locally Noetherian"
40. (24.8.11) "Fix a closed point $p \in X$" $\to$ "Fix a point $p \in X$". The scheme X may not have any closed points at all. We can, however, assume that p is closed in the fiber $\pi^{-1}(q) = X_q$. Also, "Let $R := B[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$" $\to$ "Let $R := B[x_1, \ldots, x_N]/(f_1, \ldots, f_r)$", and similarly in Figure 24.8.
41. (25.2.2) "Our base case is $m = p$: take $K^n = 0$ for $n > p$.)" $\to$ "Our base case is $m = n$; take $K^p = 0$ for $p > n$.)"
42. (25.2.5) "$J^{-1} = \ker(K^{-1} \to K^0)$" $\to$ "$J^{-1} = \operatorname{im}(K^{-1} \to K^0)$", also in the figure.
43. (25.2.9) "that $\ker \delta^p$ is finite rank locally free, and the construction of $\ker \delta^p$ commutes with any base change in a neighborhood of q" $\to$ "that $\ker \delta^p$ is finite rank locally free, and the construction of $\ker\ \delta^p$ commutes with any base change, in a neighborhood of q" and "Now $\mathrm{H}^p(K^\bullet)$ is locally free" $\to$ "Now $\mathrm{H}^p(K^\bullet)$ is locally free in a neighborhood of q" and similarly for "map of vector bundles" twice. Finally, "if and only if $\mathrm{H}^{p-1}(K^\bullet) \to \mathrm{H}^{p-1}(K^\bullet|_q)$ is surjective" $\to$ "if and only if $\kappa(q) \otimes \mathrm{H}^{p-1}(K^\bullet) \to \mathrm{H}^{p-1}(K^\bullet|_q)$ is surjective". See also the comment about 25.2.9 in Substational Corrections above.
44. (26.1.4(i)) "N has finite length" $\to$ "N has finite nonzero length". The zero module does not have $\operatorname{Supp} N = \{\mathfrak{m}\}$.
45. (27.2.2) "(over $\mathbb{Z}$)" should be removed, since we decided to work over a field $\overline{k}$ from the beginning of the chapter. (Although, I think that it is worth developing the theory over $\mathbb{Z}$ to see, e.g., why the Fermat cubic over $\mathbb{C}$ has anything to say about lines on cubic surfaces over $\overline{\mathbb{F}}_3$.)
46. (27.2.F) "where these functions are positive" $\to$ "where these functions are nonnegative"
47. (27.3.D) "trasnversely" $\to$ "transversely"
48. (28.1.7, in the Caution) "with respect to the ideal $((t, t, t, \ldots))$" $\to$ "with respect to the ideal (t)"
49. (28.4.C) "contracting the same connected subset to a point" $\to$ "contracting the same subset to a closed point" (I don't know what it means to be an isomorphism elsewhere if the point is not closed; also, doesn't "connected" follow automatically from 28.4.B since we're assuming Y and Y' to be normal?)
50. (28.4.D) "of a morphism $\pi : X \to Y$" $\to$ "of a proper (or at least qcqs) morphism $\pi : X \to Y$"
51. (28.6.N) Projs should be replaced by $\mathbb{P}$s throughout
52. (28.7.3 and 28.7.4) "$\lim_{n\to\infty} \mathrm{H}^i(X, \mathscr{F}) \otimes_B \hat{B}$" $\to$ "$\mathrm{H}^i(X, \mathscr{F}) \otimes_B \hat{B}$"
53. (29.2.B) "$\mathrm{Mod}_{\mathscr{O}_Y}$" $\to$ "$\mathrm{Mod}_{\mathscr{O}_U}$"

3 Other Minor Suggestions for Improvement

1. (10.3.G) This has an unnecessary "affine" in the hypotheses. This makes appeal to it trickier when doing 20.1.J; this "removal of affine" step I think could be done in this exercise.
2. (10.6.D) This may be a style thing, but I find this exercise very unsatisfying and to me feels like it goes against the style of the book (which I otherwise like very much). This is because the construction $S \times_A T = \sum_n S_n \otimes_A T_n$ does not respect the results of 7.4.4 and hence feels morally wrong. (I have the same problem with the corresponding exercise in Hartshorne.) In my mind, the correct way to do this is to work with multiproj of multigraded rings (as is done in [11]), which seems a little tedious to set up, but in the case of $\mathbb{N}^m$ grading is really not that bad; I have attached

an attempt at doing this in §5.

3. (10.7.P and 28.4.B) I think we should replace "irreducible varieties" by "integral locally Noetherian schemes". This is what we need in 28.4.C, for instance.
4. (11.4.1) In the definition of proper morphisms in 11.4.1 (and Hartshorne, and the Stacks Project, and EGA actually!), the finite type (as opposed to locally of finite type) hypothesis is redundant, since a universally closed morphism of schemes is automatically quasicompact. This is an old observation by Bjorn Poonen ([12]). I just mention this because it fits very nicely with the book's philosophy of minimal definitions, and I think this result deserves to be better known than it is.
5. (After 12.2.F) ("This is true more generally when A is Noetherian...") This is in fact a consequence of 12.4.A, which contains the nontrivial direction needed for the proof ([10, Corollary 2.5.7]). Maybe this is worth including, given how close we are!
6. (12.2.G(c)) I know how to do 12.2.G(c) using 12.2.9 + 15.7.I or using 12.3.2, but not without using either. Similarly, the notion of a "cone over a closed subset of $\mathbb{P}^n_k$", as it appears in 12.3.D or 12.3.G, needs defining or waiting until we know all closed subschemes of $\mathbb{P}^n_k$. Unless we intended the reader to come up with these themselves, I think the only way to salvage the exposition without serious rearrangement is to add an IOU stating that all closed subschemes of $\mathbb{P}^n_k$ are defined by homogeneous ideals.
7. (12.3.F) The exercise 11.2.M quoted in 12.3.F only handles the open case. Perhaps there should be a statement saying "a similar argument as in 11.2.M" does the trick.
8. (13.2.F) We should require f to be nonzero. (Perhaps this is implicit when we use the word "hypersurface".)
9. (13.2.9) I think $p = 2$ is fine also.
10. (13.2.M(b)) "Suppose $\pi : X \to Y$ is closed embedding" $\to$ "Suppose $\pi : X \to Y$ is a locally closed embedding". (This fulfils the promise made in 9.5.)
11. (13.2.14) The second paragraph in the proof of 13.2.14 can be significantly simplified by an appeal to 13.2.B.
12. (13.3.B(b)) "We won't use this, so we won't prove it." I think it is worth quoting the openness of the regular locus in the Stacks project ([3, 07R2]), for instance. (I dislike statements of this sort which mention unreasonably hard, i.e. not exercise-able, statements without giving a reference...)
13. (13.3.K) I think it might make sense to move 13.3.K to before 13.3.J.
14. (Proof of 13.3.7): I do not see the reason to invoke Krull's height theorem at all, since all we are using is that the dimension is less than the embedding dimension, which is just Nakayama (12.3.K).
15. (13.4.A(a)) I find the phrase "finitely many non-smooth points" confusing (since a non-smooth point is not a thing); perhaps "smooth away from finitely many closed points" would be better?
16. (13.5.20) "Another generalization ... and proof." I do not see how this statement is a generalization of Algebraic Hartogs, which I think is fundamentally about depth. The Atiyah-Macdonald statement is a trivial consequence of the definition of a valuation ring and Lang's Lemma. For instance, it is not clear how to deduce 13.5.19 from the Atiyah-Macdonald statement.
17. (13.5.13) "By Theorem 13.5.8, Noetherian normal schemes are regular in codimension 1." $\to$ "By Theorem 13.5.8, locally Noetherian normal schemes are regular in codimension 1."
18. (Proof of 13.6.6) The first paragraph in the proof of 13.6.6 can be simplified by an appeal to exercise 13.1.M, and part (b) simplified by an appeal to the converse of Krull's Hauptidealsatz (which should be stated).
19. (14.3.K(a)) "and the rank of $\mathscr{F}$ is constant" $\to$ "and the rank of $\mathscr{F}$ is locally constant"
20. (15.2.11) As written right now, the discussion in 15.2.11 needs our classification of line bundles on projective space, which comes later (15.4.I).
21. (18.1.6) "with Y Noetherian" $\to$ "with Y locally Noetherian".
22. (18.4.7.1) This is a definition; what is the content of the statement which is the "Riemann-Roch [theorem] for coherent sheaves on a projective curve"?
23. (18.4.11) Should this say "if $i : C \hookrightarrow X$ is a *pure* one-dimensional closed reduced subscheme"? We haven't explained what to do with isolated zero-dimensional points (which I think we just ignore for the purposes of this definition, but this needs to be said.).
24. (18.5.1) It appears to me from reading Chapter 29 that we do not need X to be geometrically irreducible for the statement to be true. Removing this hypothesis makes it cleaner to invoke 18.5.1 when doing exercises such as 21.3.A and 21.5.K, where "geometrically irreducible" is not included in the hypothesis. The trace explanation and $\mathrm{H}^n(X, \omega_X) = k$ right after do use "geometrically irreducible", however.
25. (18.6.H) "as defined in 18.4.6" $\to$ "as defined in 18.4.2", and the hint could invoke 18.4.F/M, since

I think that ultimately that is the way we are invoking 18.4.B.

26. (18.7.5) In the proof, can we not just clear denominators initially in our choice of f_i' to assume that $F = 1$? This makes the proof slightly easier to follow (for me).
27. (18.8.C) I don't see why the reference for the projection formula is 18.7.E(b) and not 14.5.L(b) + 18.2.E.
28. (18.8.3) "$\mathscr{L}^{\otimes n}$ is globally generated in each of these U_p" → "$\mathscr{L}^{\otimes n}$ is globally generated at every point of each U_p".
29. (18.8.5) I think we can simplify the proof by taking only closed points p and taking $q = p$.
30. (19.1.2) I don't think the statement or proof needs "Noetherian" or even "locally Noetherian". Maybe it is also worth saying that this is a necessary and sufficient condition.
31. (19.9.4) I believe the statement in 19.9.4 is more properly called "Mordell's Theorem"; Weil's contribution was the corresponding statement over number fields. (Some folks at Cambridge are very particular about this!)
32. (19.9.E) Should this exercise require that $\operatorname{char} k \neq 2$ to again use our understanding of quadrics?
33. (19.10.D) "The partially defined map t on this affine variety" Is it clear that C^{reg} is affine? We don't seem to have shown that a curve is either projective or affine, which is what we seem to be invoking. We also don't need this for the proof fortunately.
34. (19.10.I,J): Maybe it is worth emphasizing that we are talking about line bundles on C and not $C^\circ = C \setminus \{P\}$ where P is the singular point? I was initially quite confused by the apparent contradiction that $\operatorname{Pic}(C^\circ) = 0$, so $C^\circ(k) \to \operatorname{Pic}(C^\circ)$ could not be an injection, much less an isomorphism.
35. (19.11.1) I think this should say "Consider a closed point $p \in E$." In general, the use of "point $p \in E$" throughout this section (and others) makes me slightly queasy, but I guess that is how the literature looks too.
36. (19.11.G) Why do we need to do the "line bundle over a line bundle" trick here? Can't we just figure out the right statement for vector bundles of rank ≥ 1 in 19.11.E and use the decomposition of symmetric powers of a direct sum to prove this? I find this much cleaner conceptually.
37. (20.1.5) Why do we insist on our Xs in this chapter to be reduced? This makes the base-change step in 20.1.5 slightly trickier, since we have to worry about reduced but not geometrically reduced schemes. Wouldn't it be easier if we didn't ask for any reducedness at all, which is not needed for the exposition whatsoever?
38. (20.3.A) We can remove the "k is infinite" hypothesis by an appeal to 16.2.K.
39. (21.2.G) "finite type scheme" → "locally finite type scheme"
40. (21.4.C) We haven't defined degrees or multiplicities of divisors at points. I think this is worth explaining.
41. (21.4.F) I find the language of the problem slightly imprecise. If we take "unbranched" to mean "étale" and "cover" to mean "surjective map", then the statement that the only connected étale surjective morphism $X \to \mathbb{A}_k^1$ is an isomorphism is false, as evidenced by the bug-eyed line. The problem can be made clearer by replacing the words "connected unbranched cover of $\mathbb{A}_k^1$" by "surjective finite étale morphism from connected scheme to $\mathbb{A}_k^1$".
42. (21.4.K(a)) I think it is worth explicitly acknowledging that it is not a priori obvious that the extension $K(C)^G$ is finitely generated over k, and this is a fact the reader has to prove in the course of doing this exercise.
43. (21.5.A) The hint can be improved by asking the reader to use 14.2.E instead.
44. (Right before 21.5.C) "We will connect the geometric genus to the topological genus ... soon." It seems there has been some reorganization to the structure here, and this sentence can be removed.
45. (21.5.G) Why do we insist on the field k being algebraically closed here? If we don't, then 29.4.H can directly refer to 21.5.G again, instead of asking the reader to go to 21.5.B.
46. (Proof of 21.6.1, Line −4): "if K/k is a transcendence degree n" → "if K/k is a finitely generated transcendence degree n"
47. (22.3.1) "And if $Y \to Z$ is projective and Z is quasicompact" → "And if $Y \to Z$ is projective and Z is affine or Noetherian" (that's what 17.3.I gives)
48. (22.3.2) "is an effective Cartier divisor in class $\mathscr{O}(1)$" This is confusing (to me); I usually take an effective divisor in class $\mathscr{O}(1)$ to mean a divisor D such that $\mathscr{O}(D) \cong \mathscr{O}(1)$ (e.g., a hyperplane in $\mathbb{P}^n$). Perhaps it is worth emphasizing here that this is not the case here, and that here this means that $\mathscr{O}(-E) = \mathscr{O}(1)$.
49. (22.3.4) "In the case of the blow-up of a regular subvariety of a regular variety, the blow-up turns out to be regular (see Theorem 22.3.10)" Theorem 22.3.10 deals with the smooth case; perhaps it is

worth saying that a small modification of that proof gives us this result, or changing the statement in 22.3.4 to say "smooth" instead of "regular".

50. (22.3.D(c)) "Assume further that X is a reduced closed point p" $\to$ "Assume further that Y is locally Noetherian and X is a reduced closed point p" We haven't defined what "regular point" means in the non-Noetherian setting.
51. (22.4.O) Uses 13.8.5, which we promised not to use.
52. (22.4.16) "Suppose $p_1, \ldots, p_n$ are closed points" $\to$ "Suppose $p_1, \ldots, p_n$ are points"
53. 24.1.3: The proof can tightened by removing the first paragraph and replacing it with a call to 24.1.1(ii) and 24.1.D.
54. (24.2.N, print version only) "prove a similarly statement for a similar defined" $\to$ "prove a similar statement for a similarly defined"
55. (24.3.13) "0 is a regular point of Z" $\to$ "0 is a regular closed point of Z". Is there a reason the two examples are given separately, since the first is a special case of the second by taking $Z = \operatorname{Spec} A$?
56. (24.5.J) "flat morphism of finite type k-schemes" $\to$ "flat morphism of locally finite type k-schemes". This causes 24.5.L to be an immediate consequence of 24.5.J,K.
57. (24.6.D) "Noetherian k-scheme" $\to$ "locally Noetherian k-scheme"
58. (24.8.10) "locally finitely presented" $\to$ "locally of finite type"
59. (25.1.9) "so the class of morphisms $\pi : X \to Y$ that are proper, finitely presented, and flat, with geometrically connected and geometrically reduced fibers is reasonable". We knew this already; I don't understand what the "so" is doing here. Did we mean to say: the class of lfp proper flat morphisms with fiber $h^0 = 1$ is reasonable (and a reasonable replacement for the class of proper $\mathscr{O}$-connected morphisms)?
60. (25.1.L) We need to require X to be nonempty and $n \geq 1$. Also, maybe "X connected Noetherian" $\to$ "X connected locally Noetherian"? (The statement doesn't even need that if we use the discussion in 25.2.10.)
61. (26.2.10) I think it is best to remove "pure" from the statement. The proof doesn't need it, and checking it is often a little bit of a pain.
62. (28.5.C) "proper quasicompact" $\to$ "proper"; proper implies quasicompact.
63. (28.5.4) As far as I can tell, EGA IV_3.8.12.13 does not exist.
64. (29.2.E and ff., e.g., 29.2.K, etc.) I think it would be psychologically easier if we replaced "locally free" by "finite rank locally free" everywhere, which is the only case we need. (Infinite products of $\mathscr{O}_X$-modules not preserving exactness gives me a little bit of a headache, c.f. [13], and it is besides completely besides the point here, since we willl only ever use vector bundles.)
65. (29.4) I think it is better to replace π^{-1} with π^* everywhere, or even use something like $\mathscr{E}xt^i_Y(\mathscr{O}_X, \mathscr{G})|_X$ for $\pi^{-1}\mathscr{E}xt^i_Y(\pi_*\mathscr{O}_X, \mathscr{G})$. The idea is that this Ext sheaf is (schematically) supported on X, i.e., is the pushforward of a sheaf on X, and it's a question of notation as to what that should be called. Both π^{-1} and π^* should give us the same thing, but to me π^* feels psychologically better.

4 Some Other Rambling Thoughts

1. (13.4.B) I found this exercise very hard to do fully rigorously using the tools given up to this point; my proof involves estimating dimensions of cones over nonstandard determinant varieties in multiprojective space. Is there an easier proof?
2. (15.5.G) "this is related to the fact" How are these facts related? (Is it via the Universal Coefficient Theorem and the Weil conjectures? What is the best way to state this relation more precisely?)
3. (18.7.B(a)) I do not see how to do 18.7.B(a) in a way that is not just doing 18.7.C first; in particular, I do not see how to invoke the FHHF theorem. My solution is to cover Y by affines U_i and cover $\psi^{-1}U_i$ by say V_{ij}, and then map $\mathrm{H}^*(\pi^{-1}(U_i), \mathscr{F})$ to the limit of $\prod_j \mathrm{H}^*((\pi')^{-1}V_{ij}, (\psi')^*\mathscr{F}) \to \prod_{j,j'} \cdots$, but at that point I have just solved 18.7.C. Surely, I am missing a way to invoke the FHHF theorem directly; what is it?
4. (19.2.J) I think this is a rather difficult exercise. We can reduce the problem to showing that if X is a regular, integral, projective 1-dimensional k-scheme and $x \in X$ a closed point, then $\mathscr{O}(x)$ is ample, but no numerical criteria will come to our rescue here because geometrically X can be very nonreduced or reducible. One could try to use 18.4.L, but then one would need to prove nontrivial facts about the transitivity of Galois actions. (For context, for 19.2.H, my proof involves taking irreducible components using 18.8.B, passing to an algebraic closure using 18.8.1, passing to the reduction using 18.8.A, passing to the normalization using 18.8.C, and then again to

irreducible components using 18.8.B. It suffices to compute that pullback of $\mathscr{O}(x)$ under this–which uses the slightly nontrivial (to me) fact that if X is lft irreducible over a field k, and K an algebraic extension, then *every* component of X_K surjects onto X. The reason this proof does not work for 19.2.J directly is that without the transitivity of Galois action, 18.4.L does not give me control over degrees on each component after passing to the algebraic closure.) The proof at [3, 0B5X] has a very different flavor.

5. (19.10.7) I really enjoyed working through this proof of Poncelet's porism!
   (i) Should we require $\operatorname{char} k \neq 2$ in 19.10.M?
   (ii) What does 19.10.L have to do with the rest of this section, and why is the conic called E?
   (iii) In the follow-up questions Vakil suggests after 19.10.L, I think something subtle is happening: my proof of 19.10.L uses that matrix in $\mathrm{SL}_2(\mathbb{C})$ which is non-diagonalizable, has trace ± 2; in fact, it seems to me that the obvious generalization of E to arbitrary fields is false. What is the reader supposed to find in the remark after 19.10.L?
   (iv) Following the strategy at the end of this section, I see how to prove Poncelet for arbitrary transversely intersecting smooth conics over any field (of char $\neq 2$), and to do the ellipse case over $\mathbb{R}$ (the only nontransverse intersection we have to worry about, up to real projective changes of coordinates, is when C and D are bitangent at two complex conjugate points at infinity, i.e., are concentric circles–where the statement is obvious.) I believe the statement of 19.10.8 should be true over an arbitrary field (of char $\neq 2$), even when $C \cap D$ is nontransverse; Wikipedia says this follows from the transverse case by a "limit argument", but this is not clear to me–certainly the proof from 19.10.M doesn't work. Is this fact true, and what's a good proof?
6. (24.3.Q) I think this exercise is too hard to do by hand, particularly if the reader hasn't seen Gröbner bases before, which TRS doesn't talk about. (Of course this follows by considering the reduced Gröbner basis for the lex order $m > y > x > t$, but I tried to do it "without using Gröbner bases" and failed to show uniqueness.) This might be a more subjective thing.
7. (24.7.E) I think it is somewhat fiddly to glue line bundles together. Wouldn't it be easier to use the (reduction of the) product of the two "witness" k-schemes, using that connected + k-rational point implies geometrically connected? This may be a subjective thing too.
8. (28.1.F) I am definitely missing something here, because I don't see what connection this proof has to Hensel's Lemma. The proof of this result I came up with is, as I later realized, the same as the one in [14, Prop. 4.3.26]. How does this relate to Hensel's Lemma at all?

5 Multiproj

This section has been adapted from [11].

Fix $m \in \mathbb{N}$. A submonoid $\Lambda \subset \mathbb{N}^m$ is *relevant* iff for all $\alpha \in \mathbb{N}^m$, there is an $n \in \mathbb{N}_{>0}$ and $\lambda, \mu \in \Lambda$ so that $n\alpha = \lambda - \mu$. Let R be an $\mathbb{N}^m$-graded ring. Let $\mathscr{S}$ be the set of multiplicative submonoids $S \subset R$ generated by finitely many nonzero homogeneous elements f_i such that the submonoid $\Lambda_S = \langle \deg(f_i) \rangle_i \subset \mathbb{N}^m$ is relevant. For each $S, T \in \mathscr{S}$, note that $ST \in \mathscr{S}$ as well. For $S \in \mathscr{S}$, define $R_{(S)} := (S^{-1}R)_0$, and glue as in usual Proj to obtain the scheme $\operatorname{Proj}^m R$. Note that replacing $\mathscr{S}$ by a subset $\mathscr{S}' \subset \mathscr{S}$ which is coinitial and stable under products gives us the same scheme.

Examples:

(a) $\operatorname{Proj}^0 R = \operatorname{Spec} R$
(b) $\operatorname{Proj}^1 R = \operatorname{Proj} R$,
(c) $\operatorname{Proj}^2 \mathbb{C}[x_0, x_1, y_0, y_1] = \mathbb{P}^1_{\mathbb{C}} \times_{\operatorname{Spec} \mathbb{C}} \mathbb{P}^1_{\mathbb{C}}$,

More generally, we have $\operatorname{Proj}^{m+n}(R \otimes_A S) = \operatorname{Proj}^m R \times_{\operatorname{Spec} A} \operatorname{Proj}^n S$. For example:

(d) $\operatorname{Proj}(R \otimes_A B) = \operatorname{Proj}(R) \times_{\operatorname{Spec} A} \operatorname{Spec} B$.
(e) When $m = n = 1$ and $R \times_A S = (R \otimes_A S)_\Delta = \bigoplus_{n \in \mathbb{N}} R_n \otimes_A S_n$, then the inclusion $R \times_A S \hookrightarrow R \otimes_A S$ induces an isomorphism $\operatorname{Proj}^2(R \otimes_A S) \to \operatorname{Proj}^1(R \times_A S)$. But there are many other "ways" to go to (∞, ∞); for any suitably large single-direction submonoid $\Lambda \subset \mathbb{N}^2$ (here meaning that it eventually is periodic), we will get that $\operatorname{Proj}^2(R \otimes_A S) \cong \operatorname{Proj}^\Lambda((R \otimes_A S)_\Lambda)$.

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