The Atiyah-Singer Index Theorem and Almost Complex Spheres

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Motivating Question

Q.: When is a manifold X the underlying space of a complex manifold?

For this, X needs to be
(1) even-dimensional,
(2) smooth(able), and
(3) orientable.

Not sufficient! Need more refined criteria.

Almost Complex Structures

If X is a complex manifold, then its (smooth) tangent bundle $\mathrm{T}X$ has the structure of a complex vector bundle:

$$\mathrm{T}X \cong \mathrm{T}^{1,0}X|_{\mathbb{R}}.$$

Definition

A smooth manifold X is **almost complex** if its tangent bundle $\mathrm{T}X$ is a $\mathbb{C}$-VB.

This is a nontrivial necessary condition.

Fact: every almost complex manifold is even-dimensional and orientable.

Sufficient when $\dim X = 2$, but not when $\dim X \geq 4$ (e.g. $\mathbb{CP}^2 \# \mathbb{CP}^2 \# \mathbb{CP}^2$).

Spheres I

When does S^n admit an almost complex structure (ACS)?

Well, n has to be even. For small even n, we have

(1) S^0,
(2) S^2, because $S^2 \cong \mathbb{CP}^1$ via stereographic projection or $S^2 \subset \mathbb{R}^3 = \operatorname{Im} \mathbb{H}$,
(3) **not** S^4 (Ehresmann-Hopf, 1949), and
(4) S^6, because $S^6 \subset \mathbb{R}^7 = \operatorname{Im} \mathbb{O}$ (Kirchoff, 1947).

What about others?

Theorem (Borel-Serre, 1953)

If $n \neq 0, 2, 6$, then S^n does not admit an ACS (w.r.t. any smooth structure).

Index Theory: Spin Manifolds, Spinor Bundles, Dirac Operator

A smooth manifold X is **orientable** iff $w_1(X) = 0$.

Definition

A smooth manifold X is said to be **spin** if $w_1(X) = w_2(X) = 0$.

Given a metric on X, this amounts to a lift of $\mathrm{SO}(X)$ to a principal Spin_n-bundle, where $\mathrm{Spin}_n \to \mathrm{SO}_n$ is a double cover (where $n := \dim X$).

When n is even and X is spin, the spin representations $\mathcal{S}_n^\pm$ of Spin_n give rise to $\mathbb{C}$-VB's on X called **spinor bundles**, often denoted $\mathcal{S}^\pm(X)$.

There is a first order differential operator

$$\not{D}^+ : \mathcal{S}^+(X) \to \mathcal{S}^-(X)$$

called the **Atiyah-Singer-Dirac operator**; motivated from physics.

The Index Theorem

If $E \to X$ is a $\mathbb{C}$-VB, there is a **twisted Atiyah-Singer-Dirac operator**

$$\not{D}_E^+ : \mathcal{S}^+(X) \otimes E \to \mathcal{S}^-(X) \otimes E.$$

This operator is **elliptic**, so that if X is closed, then the kernel and cokernel of $\not{D}_E^+$ are finite dimensional, and the **index** of $\not{D}_E^+$ is defined as

$$\operatorname{ind} \not{D}_E^+ = \dim \ker \not{D}_E^+ - \dim \operatorname{coker} \not{D}_E^+ .$$

The Atiyah-Singer Index Theorem allows us to compute $\operatorname{ind} \not{D}_E^+$ using topological information about E and X. Precisely, we have

Theorem (Atiyah-Singer, 1963)

If X is a closed even-dimensional spin manifold and $E \to X$ a $\mathbb{C}$-VB, then

$$\operatorname{ind} \not{D}_E^+ = \int_X \operatorname{ch} E \cdot \hat{\mathrm{A}}(X).$$

In particular, the quantity on the right is an integer.

Spheres II

Now we are ready to study ACSs on S^{2n} for $n \geq 1$.

Lemma

If $E \to S^{2n}$ *is a* $\mathbb{C}$*-VB,* $n \geq 1$*, then* $(n-1)!$ *divides* $c_n(E) \in \mathrm{H}^{2n}(S^{2n}; \mathbb{Z}) \cong \mathbb{Z}$*.*

Proof Sketch.

Clear for $n = 1$, so assume $n \geq 2$.

(1) S^{2n} is a closed spin manifold: $w_i(X) \in \mathrm{H}^i(S^{2n}; \mathbb{Z}/2) = 0$ for $i = 1, 2$.

(2) By ASIT, $\mathrm{ind}\, \not{D}_E^+ = \int_{S^{2n}} \mathrm{ch}\, E \cdot \hat{\mathrm{A}}(S^{2n})$.

(3) S^{2n} is stably parallelizable $\Rightarrow \hat{\mathrm{A}}(S^{2n}) = 1$.

(4) $\mathrm{H}^j(S^{2n}) = 0$ for $1 \leq j \leq 2n-1$ gives $\mathrm{ch}_n E = \frac{(-1)^{n-1}}{(n-1)!} c_n(E)$.

(5) Therefore, ASIT gives

$$c_n(E) = \pm (n-1)! \cdot \mathrm{ind}\, \not{D}_E^+ .$$

□

Spheres III

We are now ready to prove

Theorem (Borel-Serre, 1953)

If $n \neq 0, 2, 6$, then S^n is not almost complex (w.r.t any smooth structure).

Proof Sketch.

(1) If S^{2n} AC, then by Lemma, $c_n(\mathrm{T}S^{2n})$ is divisible by $(n-1)!$.
(2) By Chern-Gauss-Bonnet, $c_n(\mathrm{T}S^{2n}) = e(\mathrm{T}S^{2n}) = \chi(S^{2n}) = 2$.
(3) Therefore, S^{2n} AC $\Rightarrow (n-1)! \mid 2 \Rightarrow n \leq 3$.
(4) Handle S^4 separately: $c_1(E)^2 = 2\chi(X) + 3\sigma(X)$.

□

Last Remarks

Other proofs:

(1) (Borel-Serre, 1953) Using mod p Steenrod power operations, and

(2) (Kirchoff, 1947) S^n AC $\Rightarrow S^{n+1}$ parallelizable $+$
(Hirzebruch, Kervaire, Bott-Milnor, Adams, 1958) only S^1, S^3, S^7.

Other applications of the ASIT:

(1) The $\hat{A}$-genus of a closed spin manifold X is an integer.
It is even if $\dim X \equiv 4 \pmod 8$.

(2) (Rochlin, 1952) The signature of a closed smooth spin 4-fold is div. by 16.

(3) (Freedman, 1982) There is a non-smoothable topological 4-fold $X(E_8)$.

(4) Other non-ACSs: $\mathbb{HP}^n$ (Hirzebruch, 1953; Massey, 1962) and $\#^{2m}\mathbb{CP}^{2n}$.

(5) Much, much more!

Open problems:

(1) Does S^6 admit a complex structure?

(2) $\#^{2m+1}\mathbb{CP}^2, m \geq 1$, not AC (Van de Ven, 1966; Enriques-Kodaira 1968).
What about $\#^{2m+1}\mathbb{CP}^{2n}$ for $m \geq 1$ and $n \geq 2$?

Thanks!

Thank you!