1.6 26/06/23 - Review of Projective Geometry IV: Wrapping up Smoothness, and Bézout's Theorem

Let me say a few more things about smoothness first, and let's go over a few examples. Then we'll talk about Bézout's Theorem. After this, starting next time, we'll start talking about cubic curves and group laws on them.

1.6.A Wrapping up Smoothness

Let's do a few examples.

Example 1.6.1. Let's study the example of the curve $C = \mathbb{V}(Y^2Z - X^3 - Z^3) \subset \mathbb{P}^2_K$ we saw in 1.3.12. For simplicity, we will assume that $\operatorname{ch} K \neq 2, 3$; we will figure out what happens in these characteristics later.

First, let's study what happens in the two affine patches.

(a) In the patch $Z \neq 0$ with coordinates $(x, y) = (X/Z, Y/Z)$, we have the curve $\mathbb{V}(y^2 - x^3 - 1)$. This equation has partials $-3x^2, 2y$. If $\operatorname{ch} K \neq 2, 3$, then any common solution has $x = y = 0$, but the point $(0, 0)$ doesn't lie on the curve. This proves that this patch is smooth. The tangent line at say $R = (2, 3)$ looks like $-2x + y + 1 = 0$.

(b) In the patch $Y = 0$ with coordinates $(t, s) = (X/Y, Z/Y)$, we have the curve $\mathbb{V}(s - t^3 - s^3)$. This equation has partials $-3t^2, 1 - 3s^2$. Again, there are no common solutions, and this shows that this patch is smooth as well. The tangent line at $R = (2/3, 1/3)$ then is $-2t + 1 + s = 0$.

Therefore, we have shown that the curve C is smooth, at least when $\operatorname{ch} K \neq 2, 3$.

We could do this "globally" instead of looking at affine patches. To study its singular points, for $F = Y^2Z - X^3 - Z^3$, we have partial derivatives

$$-3X^2, 2YZ, Y^2 + 3Z^2.$$

Again, since we are assuming $\operatorname{ch} K \neq 2, 3$, this forces $X = 0$, and either $Y = 0$ or $Z = 0$, and either forces the other by $Y^2 + 3Z^2 = 0$. Therefore, there are no solutions, i.e., the curve is smooth. Finally, the tangent line at $R = [2 : 3 : 1]$ is $\mathbb{T}_R(C) = \mathbb{V}(-2X + Y + Z)$.

Do you see how the "projective" and "affine computations" ((a) and (b)) relate to each other? You should now be in a good position to attempt Exercise 2.2.6.

Let's see what happens with the same curve in characteristic 3; I'll leave characteristic 2 to you below.

Example 1.6.2. Let K be a field with $\operatorname{ch} K = 3$, and $C = \mathbb{V}(Y^2Z - X^3 - Z^3) \subset \mathbb{P}^2_K$. As before, in the affine patches, we get:

(a) In the patch $Z \neq 0$, we get the singularity $(x, y) = (-1, 0)$.

(b) In the patch $Y \neq 0$, we see that there are no singularities, i.e., the patch $Y \neq 0$ is smooth.

Globally, we see that a singularity would have $Y = 0$, which forces it to be the point $[-1 : 0 : 1]$, which is indeed a singularity. Note that this is not in the affine patch $Y \neq 0$. In particular, this projective curve has a unique singular point, namely $[-1 : 0 : 1] \in C(K)$.

Exercise 1.6.3. Figure out what happens when $\operatorname{ch} K = 2$. How many singular points do we have?

As one more example, let's classify all smooth conics in the plane.

> **Proposition 1.6.4.** Let K be a field, and $C \subset \mathbb{P}^2_K$ be a smooth conic with a K-rational point $P \in C(K)$. Then there is a projective change of coordinates taking C to $\mathbb{V}(Y^2 - XZ) \subset \mathbb{P}^2_K$. In particular, C is the (isomorphic) image of a quadratic map $\mathbb{P}^1_K \to \mathbb{P}^2_K$, i.e., can be parametrized via map of the form $[S : T] \mapsto [A(S, T) : B(S, T) : C(S, T)]$ for some homogeneous quadratic polynomials $A, B, C \in K[S, T]$ with no common roots (over an algebraic closure).

Remark 1.6.5. Note that the result of this example is true in any characteristic (including $\operatorname{ch} K = 2!$), and not true if we assume only that $C \subset \mathbb{P}^2_K$ is a smooth conic, since C might not have any K-points at all! A simple example is given by $\mathbb{V}(X^2 + Y^2 + Z^2) \subset \mathbb{P}^2_{\mathbb{R}}$, which is isomorphic to $\mathbb{V}(Y^2 - XZ)$ over $\mathbb{C}$ but not over $\mathbb{R}$.

Proof. The second statement follows from the first, since the conic $\mathbb{V}(Y^2 - XZ)$ can be parametrized as $[S : T] \mapsto [S^2 : ST : T^2]$. Therefore, we just have to figure out the right change of coordinates.

Here's the proof idea: take the point P and the tangent line L to P. Take some *other* line M through P. This M will intersect C in another K-point $Q \in C(K)$ by a baby version of Bézout's Theorem. Now take the tangent line N to C through Q. This is different than L (it doesn't contain P), and so intersects L in a third K-point R. Now take the projective change of coordinates sending (P, Q, R) to $([0 : 0 : 1], [1 : 0 : 0], [0 : 1 : 0])$, and note that this works, up to a minor rescaling of one coordinate.

Let's work this out explicitly in practice. We can certainly take a change of coordinates sending P to $[0 : 0 : 1]$ and the tangent line $\mathbb{T}_P C$ (which exists since we assumed C is smooth!) to $L = \mathbb{V}(X)$. In these coordinates, C looks like

$$aX^2 + bXY + cY^2 + dXZ + eYZ + fZ^2 = 0.$$

Since $P = [0 : 0 : 1]$ lies on C, we see that $f = 0$. Since $\mathbb{T}_P(C) = \mathbb{V}(X)$, we see that $d \neq 0$ and $e = 0$. Therefore, the equation really has the form

$$aX^2 + bXY + cY^2 + dXZ = 0. \tag{1.6.6}$$

Now take the other line M to be $M = \mathbb{V}(Y)$. To figure out its intersection with C, we set $Y = 0$ in 1.6.6 to get $X(aX + dZ) = 0$, so the other point of intersection Q is $Q = [d : 0 : -a]$. This fulfils our promise of finding a different point Q. Now you can check that $P \notin \mathbb{T}_Q C$ (do this!), and this allows us to do what we promised above: take R to be the intersection of L and $\mathbb{T}_Q(C)$, and now note that P, Q, R are non-collinear (check!), which allows us to send them to $[0 : 0 : 1], [1 : 0 : 0], [0 : 1 : 0]$. When we do this, we still get an equation of the form (1.6.6), but now since Q lies on it, we have $a = 0$. Finally, the fact that $\mathbb{T}_Q(C) = \mathbb{V}(Z)$ implies that $b = 0$ as well. This leaves us with an equation of the form $cY^2 + dXZ = 0$. Since C is smooth, neither c nor d can be zero, and hence up to rescaling Z we get an equation of the desired form. ■

Exercise 1.6.7. Fill in the gaps in the above proof: show that $P \notin \mathbb{T}_Q C$, and that P, Q, and R are not collinear.

Note that 1.6.4 gives us another way to parametrize smooth conics with a rational point in the plane: figure out the change of coordinates explicitly, and then see how the parametrization $[S^2 : ST : T^2]$ changes under it.

Exercise 1.6.8. Apply the above technique to give a second solution to Exercise 2.1.6(a).

That's all we will say about smoothness in general for now, but it's an idea we'll keep revisiting.

1.6.B Bézout's Theorem

Let's first ask a very natural question: given a field and two curves $C, D \subset \mathbb{A}^2_K$, how many points can $C(K) \cap D(K)$ have?

Are there always finitely many points? No. A silly example is when $C = D$. For a less silly example, consider $C = \mathbb{V}(x(x^2 + y^2 - 1))$ and $D = \mathbb{V}(x(y - x^2))$. The problem, of course, is that C and D contain a common "component" of the y-axis $\mathbb{V}(x)$. We'll define components formally soon, but let's pretend we understand what they are. It is then at least believable that there can only be finitely many points of intersection.

How many? Let's consider a few examples. If we intersect $C = \mathbb{V}(x^2 + y^2 - 1)$ with the line $D = \mathbb{V}(x - 2)$. Well, there are no solutions over $\mathbb{Q}$ or $\mathbb{R}$, but we know this is a fault of these fields (c.f. 1.4.3). This suggests we work over an algebraically closed field such as $K = \mathbb{C}$, where we would expect 2 solutions. Over such a field, this example certainly works, and so does an example of the form $C = \mathbb{V}(y - f(x))$ for some polynomial $f(x) \in K[x]$ and the x-axis $D = \mathbb{V}(y)$, where we expect $\deg f$ intersection points. Similarly, we would expect a cubic and a conic to intersect in $2 \cdot 3 = 6$ points. More generally, we would expect curves of degrees d and e to intersect in de points. However, even over such a field, we could take $D = \mathbb{V}(x - 1)$, which gives us just one point $(1, 0)$. A similar example comes from taking $C = \mathbb{V}(y - f(x))$ for some polynomial $f(x) \in K[x]$ and $D = \mathbb{V}(y)$, the x-axis, when $f(x)$ has a repeated root. This suggests we should count things with multiplicites.

Even when you try to do that, sometimes you run into problems. The simplest example of this is when you take $C = \mathbb{V}(x)$ and $D = \mathbb{V}(x - 1)$. In this case, there are no points because the lines are parallel. A more subtle example is when we take $C = \mathbb{V}(y - x^2)$ and $D = \mathbb{V}(sy - x)$ for various s. Again for $s \neq 0$, there are two points of intersection $(0, 0)$ and $(1/s, 1/s^2)$; what happens as $s \to 0$ is that the second point "escapes to infinity", as in 1.3.2. This suggests that we should look at the projective plane as before. It is a miracle that nothing else goes wrong:

Theorem 1.6.9 (Bézout). Let K be a field, $C, D \subset \mathbb{P}^2_K$ be curves with no common component. Then for any algebraically closed field $\Omega \supset K$, the intersection $C(\Omega) \cap D(\Omega)$ consists of precisely $(\deg C) \cdot (\deg D)$ points, when counted with multiplicites.

This is named after the same guy, the 18$^{\text{th}}$-century French mathematician Étienne Bézout, after whom the identity involving greatest common divisors is named. The history of the proof of this result is quite intriguing, and I will refer you to the corresponding Wikipedia page to read more if you're interested. The result itself (1.6.9) is somewhat nontrivial, and belongs properly to a course on algebraic geometry. Part of the difficulty lies in coming up with a good definition of the intersection multiplicities at points, which is a subtle notion in general. We won't prove it; if you are interested in seeing a full proof, I can refer you to [4, 1.14.1] or [5, §5.3].

Since we won't prove this result, we won't use it either. The only case we will really need is when one of C and D, say D, is a line; that is the case we will prove next time. In the course of proving this result, we will see how to make each of the terms in the statement of the theorem precise.