1.5 26/06/22 - Review of Projective Geometry III: Smoothness

The next thing about curves we will need is the notion of smoothness. Before we give a formal definition, let's consider some examples.

Example 1.5.1. Which of the following curves is "smooth"? (a) $3x+5y=1$ (b) $x^2+y^2=1$ (c) $x^2-y^2=1$ (d) $x^2-y^2=0$ (e) $y^2=x^3+1$ (f) $y^2=x^3+x$ (g) $y^2=x^3+x^2$ (h) $y^2=x^3$.

Hopefully, the following definition feels reasonably motivated:

Definition 1.5.2. Let K be a field, and $f(x,y) \in K[x,y]$ be nonzero. Let $C = \mathbb{V}(f) \subset \mathbb{A}^2_K$ be the affine curve defined by f.

(a) Given a field extension L/K and a point $P=(x_0,y_0) \in C(L)$, we say that P is a *singular point* of C iff it is simultaneously a solution of

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$$

in $\mathbb{A}^2(L)$, and a *smooth point* otherwise. When P is a smooth point, we define the *tangent line* $\mathrm{T}_P C$ to C at P to be the line

$$\mathrm{T}_P C := \mathbb{V}\left(\left.\frac{\partial f}{\partial x}\right|_P (x-x_0) + \left.\frac{\partial f}{\partial y}\right|_P (y-y_0)\right) \subset \mathbb{A}^2_L.$$

(b) The curve C is said to be *smooth*[a] iff for every field L/K, every point of $C(L)$ is smooth.

[a] Also known as *geometrically regular*.

Make sure this agrees with your intuitive definition of the tangent line to a curve at a point!

Exercise 1.5.3. Let $\ell \subset \mathbb{A}^2_K$ be an affine line. Show that for any $P \in \ell(K)$, we have that $\mathrm{T}_P \ell = \ell$.

Exercise 1.5.4. Figure out the "singular locus" of each of the curves from 1.5.1. Does the characteristic of the base field matter? What happens for the curve $x^2+y^2=0$ over $K=\mathbb{R}$?

Unimportant Remark 1.5.5. Why do we need to insist on checking all extensions L/K? We don't need to; a version of Hilbert's Nullstellensatz tells us that it suffices to check one: any algebraically closed field Ω (e.g., $\Omega = \overline{K}$) containing K (e.g., if $K=\mathbb{Q}$, it suffices to look at $\Omega=\mathbb{C}$, i.e., $\mathbb{C}$-valued points). In other words, if every point of $C(\Omega)$ is a smooth point, then C is smooth. (This needs an argument, which I will not give here. Ask me if you're interested.) Here's an example to explain why we can't just check K-points: take $K=\mathbb{F}_3(t)$ and $f(x,y) := y^2 - x^3 + t \in K[x,y]$. The partial derivatives here are 0 and $2y$ respectively. We see that the "singular locus" is cut out by the equations $y = x^3 - t = 0$, which has no K-point, but is nonetheless nonempty: it contains the L-point $(t^{1/3},0)$ where $L := \mathbb{F}_3(t^{1/3})$. Such examples are pathologies we wish to avoid by taking our definition above, which is consistent with that of modern algebraic geometry. This example will return later when we discuss singularities of plane cubics.

When dealing with projective curves, we could either work with affine patches, or give a global definition. If we take the former approach, we would then have to justify that the choice of affine patch doesn't matter. This can be done, but the global definition is probably easier:

Definition 1.5.6 (Smoothness). Let K be a field, $F \in K[X,Y,Z]$ a nonzero homogeneous polynomial, and $D = \mathbb{V}(F) \subset \mathbb{P}^2_K$ the plane curve with defining equation F.

(a) Given a field extension L/K and a point $P \in D(L)$, we say that P is a *singular point* of D iff it is simultaneously a solution of

$$F = \frac{\partial F}{\partial X} = \frac{\partial F}{\partial Y} = \frac{\partial F}{\partial Z} = 0,$$

in $\mathbb{P}^2(L)$, and a *smooth point* otherwise. When P is a smooth point, we define the *(projective) tangent line* $\mathbb{T}_P D$ *to* D *at* P to be the line

$$\mathbb{T}_P D := \mathbb{V}\left(\frac{\partial F}{\partial X}\bigg|_P X + \frac{\partial F}{\partial Y}\bigg|_P Y + \frac{\partial F}{\partial Z}\bigg|_P Z\right) \subset \mathbb{P}^2_L.$$

Here the notation means: pick a representative $(X, Y, Z) \in L^3 \smallsetminus \{(0, 0, 0)\}$ for $P = [X : Y : Z]$, and evaluate the partials simultaneously on that representative, noting that the triple of partials only changes up to simultaneous scaling by a different choice of representative, from which it follows that the corresponding line is well-defined.

(b) The curve D is said to be *smooth*[a] iff for every field L/K, every point of $D(L)$ is smooth.

[a] Also known as *geometrically regular*.

At this point, we need to insert an exercise to make sure that these two definitions really are the same. In particular, the projective definition behaves well under taking affine patches. Because this is important enough, I have isolated this and added this to Exercise Sheet 2 (2.2.6(a)). The advantage of the projective definition is that it is highly symmetrical, and in particular stable under projective changes of coordinates (2.2.6(b)).

Unimportant Remark 1.5.7. If $F \in K[X, Y, Z]$ is homogeneous of degree $d \in \mathbb{Z}_{\geq 0}$, then *Euler's formula* tells us that

$$X\frac{\partial F}{\partial X} + Y\frac{\partial F}{\partial Y} + Z\frac{\partial F}{\partial Z} = d \cdot F.$$

(Prove this! See 2.1.11.) In particular, if $\operatorname{ch} K \nmid d$, then the locus given by the vanishing of partials is automatically contained in the vanishing locus of F. The example of 1.5.5 shows that this is not true if $\operatorname{ch} K \mid d$ in general.

Example 1.5.8. If we take $K = \mathbb{Q}$ and $F = Y^2Z - X^3$, then for any field extension $L/\mathbb{Q}$, there is a unique singular point of $D(L)$, namely $[0 : 0 : 1]$. The tangent line to D at the $\mathbb{Q}$-point $P = [1 : 1 : 1]$ is $\mathbb{V}(-3X + 2Y + Z) \subset \mathbb{P}^2_{\mathbb{Q}}$. In affine coordinates in the usual patch, this is saying that the curve $y^2 = x^3$ is singular precisely at $(0, 0)$ and has tangent line $-3x + 2y + 1 = 0$ at $(1, 1)$.

Next time, we will see how to apply these ideas in practice, and work through a few examples.