1.4 25/06/18: Review of Projective Geometry II: Projective Changes of Coordinates

Perhaps I didn't do a great job of explaining projective space last time, so let me try again.

1.4.A More on Curves and Homogenization

Let us review some things from last time. Given a field K, we met the *affine plane* $\mathbb{A}^2(K)$ over K, which is just the good old plane: it consists of all points (x, y) where $x, y \in K$:

$$\mathbb{A}^2(K) = \{(x, y) : x, y \in K\}.$$

Note that for any field extension L/K, we have a natural inclusion $\mathbb{A}^2(K) \subset \mathbb{A}^2(L)$.

Given a nonzero $f \in K[x, y]$, we can look at its *vanishing locus* in $\mathbb{A}^2(K)$, which is just

$$\{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}.$$

This is what we want to call a curve. Note that for any $\lambda \in K^\times$, the polynomials f and λf have the same vanishing loci; they define the same *curve*. This motivates the algebraic

Definition 1.4.1.

- An *affine curve* is an equivalence class of nonzero $f \in K[x, y]$ under the equivalence relation $f \sim \lambda f$ for nonzero $f \in K[x, y]$ and $\lambda \in K^\times$.

The curve defined by a nonzero f will be denoted $C := \mathbb{V}(f)$, so that $\mathbb{V}(f) = \mathbb{V}(\lambda f)$ for all $\lambda \in K^\times$.

- Such an f as above is called *a defining equation* for the curve C, and the *degree of the curve* C is defined to be the degree of any defining equation for C.

A(n) (affine) curve of degree one (resp. two, three, four, etc.) is called a(n) (affine) *line* (resp. *conic*, *cubic*, *quartic*, etc.).

Given such an f and the associated curve $C = \mathbb{V}(f) \subset \mathbb{A}^2_K$, we can then look at its K-points

$$C(K) := \{(x, y) \in \mathbb{A}^2(K) : f(x, y) = 0\}.$$

The reason for keeping this notation separate is because given a polynomial $f \in K[x, y]$, we would like to study its solutions over any field extension L/K. For instance, K could be $\mathbb{Q}$, and L could be $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{Q}_p$ or some number field like $\mathbb{Q}[i]$ or $\mathbb{Q}[\sqrt{5}]$. The above notation easily lets us do that, so we get a natural inclusion $\mathbb{A}^2(K) \subset \mathbb{A}^2(L)$ and $C(K) \subset C(L)$.

Example 1.4.2. Let $K = \mathbb{Q}$, let $f(x, y) = x^2 + y^2 - 1 \in \mathbb{Q}[x, y]$, and let $C := \mathbb{V}(f)$. Then $C(\mathbb{Q}) \subset C(\mathbb{R}) \subset C(\mathbb{C})$. We have that $(1, 0) \in C(\mathbb{Q})$, that $(1/\sqrt{2}, 1/\sqrt{2}) \in C(\mathbb{R}) \smallsetminus C(\mathbb{Q})$, and $(i, \sqrt{2}) \in C(\mathbb{C}) \smallsetminus C(\mathbb{R})$.

This algebraic definition also allows us to work with curves that have no points over the base field.

Example 1.4.3. Let $K = \mathbb{Q}$, let $f(x, y) := x^2 + y^2 + 1 \in \mathbb{Q}[x, y]$, and let $C := \mathbb{V}(f)$. Then $C(\mathbb{Q}) = C(\mathbb{R}) = \emptyset$ but $C(\mathbb{C}) \neq \emptyset$, because for instance $(0, i) \in C(\mathbb{C})$. In particular, C is not "empty"; it's just that $\mathbb{R}$ is not a good enough field to see any points on it.

Exercise 1.4.4. Let K be an algebraically closed field. Then for any nonzero $f \in K[x, y]$ of positive degree, if we set $C := \mathbb{V}(f)$, then $C(K) \subset \mathbb{A}^2(K)$ is nonempty. (Note that of course this is not true if K is not algebraically closed, as we have seen above in 1.4.3.)

The abstract curve $C = \mathbb{V}(f) \subset \mathbb{A}^2_K$ arising from this algebraic definition encodes this data of points in every field extension L/K.

Unimportant Remark 1.4.5. The objects $\mathbb{A}^2_K$ and $\mathbb{A}^2(K)$ are slightly different; the latter is the set of K-points of the former (and is probably more properly written $\mathbb{A}^2_K(K)$). The first is a "formal object" that records the data of all $\mathbb{A}^2(L)$ for all extensions L/K. The same goes for the curve C and its L-points $C(L)$ for any L/K. The subscript K in the formal object reminds us what field we are doing the algebra; for instance, if $K = \mathbb{F}_3$ and $C = \mathbb{V}((1 \mod 3)y + (2 \mod 3)x + (1 \mod 3)) \subset \mathbb{A}^2_{\mathbb{F}_3}$, then the subscript reminds us that it makes no sense to talk about $C(\mathbb{Q})$, although it makes perfect sense to talk about $C(\mathbb{F}_3)$ or more generally $C(L)$ for any $L/\mathbb{F}_3$. (For the cognoscenti: $\mathbb{A}^2_K := \operatorname{Spec} K[x, y]$ and $C := \operatorname{Spec} K[x, y]/(f)$.)

Remark 1.4.6. One strange consequence of the above definition is that sometimes you can have two curves $C, C' \subset \mathbb{A}^2_K$ such that for all fields L/K we have $C(L) = C'(L)$ even though $C \neq C'$. For instance, C and C' might even have different degrees, e.g., if $C = \mathbb{V}(x)$ and $C' = \mathbb{V}(x^2)$. Why do you think this is a good thing? Is it?

More or less the same things as above can be done in the projective plane. As above,

$$\mathbb{P}^2(K) = \{[X : Y : Z] : X, Y, Z \in K \text{ not all zero}\}/\sim$$

where the $\sim$ means that we identify points $[X : Y : Z]$ and $[\lambda X : \lambda Y : \lambda Z]$ for all $\lambda \in K^\times$. As before, for each L/K, we have $\mathbb{P}^2(K) \subset \mathbb{P}^2(L)$. Finally, for any K, the affine space $\mathbb{A}^2(K)$ sits inside $\mathbb{P}^2(K)$ by sending (x, y) to $[x : y : 1]$, with the complement being the "line at infinity" given by $Z = 0$.

Given a nonzero *homogeneous* $F \in K[X, Y, Z]$, we can look at its *vanishing locus* in $\mathbb{P}^2(K)$, which is just

$$\{[X : Y : Z] \in \mathbb{P}^2(K) : F(X, Y, Z) = 0\}.$$

As before, we have

Definition 1.4.7.

- A *projective curve* is an equivalence class of nonzero homogeneous $F \in K[X, Y, Z]$ under the equivalence relation $F \sim \lambda F$ for nonzero homogeneous $F \in K[X, Y, Z]$ and $\lambda \in K^\times$.

The curve defined by a nonzero F will be denoted $D := \mathbb{V}(F)$, so that $\mathbb{V}(F) = \mathbb{V}(\lambda F)$ for all $\lambda \in K^\times$.

- Such an F as above is called *a defining equation* for the curve D, and the *degree of the curve* D is defined to be the degree of any defining equation for D.[a]

[a]For the cognoscenti: D stands for "divisor".

As before, we have projective lines, conics, cubics, quartics, etc., and as before for any $D = \mathbb{V}(F) \subset \mathbb{P}^2_K$ it makes sense to talk about $D(L)$ for any L/K, with $D(K) \subset D(L)$.

Example 1.4.9. A projective line has the form $D = \mathbb{V}(aX + bY + cZ)$ for $a, b, c \in K$ not all zero, and where we may simultaneously scale a, b, and c. In particular, the set of all lines in $\mathbb{P}^2_K$ it itself a $\mathbb{P}^2(K)$, called the *dual projective plane*. (This duality has some lots of interesting consequences.) In general, this simultaneous scaling business feels awfully similar to what we have done above in the definition of projective spaces itself. This is no coincidence. The "space" of all curves of a given degree $d \in \mathbb{Z}_{\geq 0}$ is itself a projective space $\mathbb{P}^N(K)$ for some N (which?).

Remark 1.4.10. Doing the same procedure as above for arbitrary n gives us a *hypersurface* $D = \mathbb{V}(F) \subset \mathbb{P}^n_K$ associated to any nonzero homogenous $F \in K[X_0, \ldots, X_n]$. When $n = 1$, for any field extension L/K, the set $D(L)$ is then finite and counts the number of roots–including those at infinity–of a (homogenized) single-variable equation in L (do you see why?). For example, if $K = \mathbb{Q}$ and $F = S^2 - 5ST + 5T^2$, then $D(\mathbb{Q}) = D(\mathbb{R}) = \emptyset$ but $D(\mathbb{C})$ consists of two points (c.f. 1.1.1(b)(i)).

Finally, as before we have the notions of homogenization and dehomogenization. We have that the locus $Z \neq 0$ (i.e., the complement of the line $\mathbb{V}(Z)$ at infinity) in $\mathbb{P}^2_K$ is naturally an $\mathbb{A}^2_K$:

$$\mathbb{A}^2_K = \{Z \neq 0\} \subset \mathbb{P}^2_K.$$

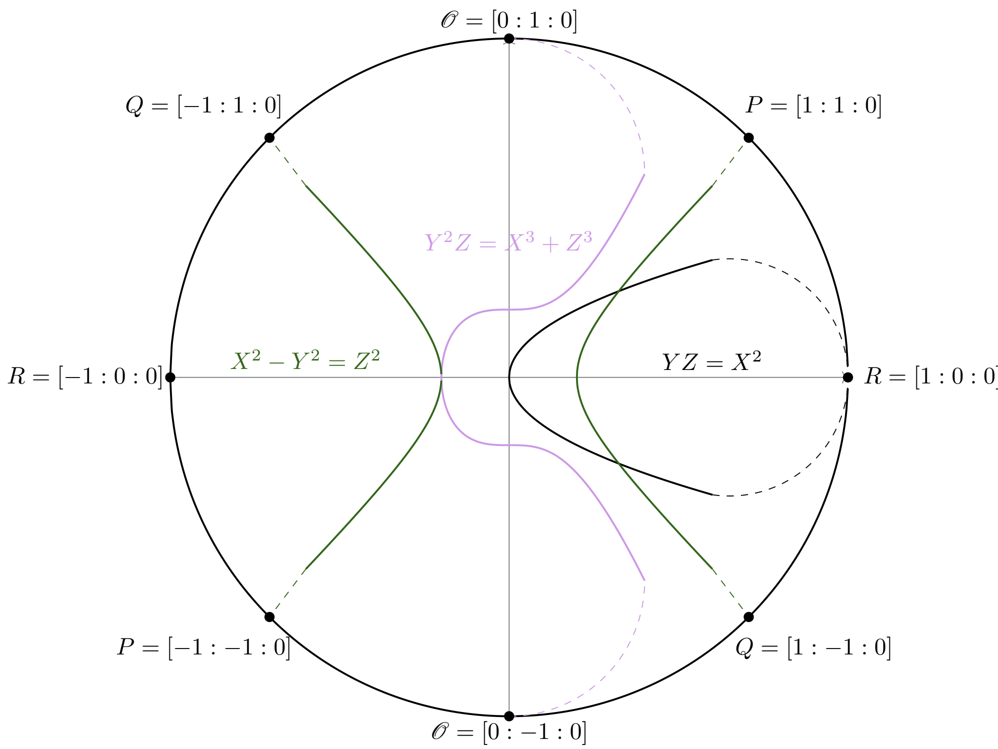

Figure 1.4.8: Some curves in the projective plane and their points at infinity.

For a projective curve $D = \mathbb{V}(F) \subset \mathbb{P}^2_K$, the intersection

$$C := D^\circ := D \cap \mathbb{A}^2_K = \mathbb{V}(f) \subset \mathbb{A}^2_K,$$

where $f(x,y) := F(x,y,1) \in K[x,y]$. This is the dehomogenization with respect to Z. Conversely, given an $f(x,y) \in K[x,y]$ of degree $d \in \mathbb{Z}_{\geq 1}$ and corresponding affine curve $C = \mathbb{V}(f)$, its *projective completion* is the projective curve

$$D := \overline{C} = \mathbb{V}(F) \subset \mathbb{P}^2_K,$$

where $F := Z^d f(X/Z, Y/Z) \in K[X,Y,Z]$. These two operations are basically inverses, but this is not literally true. For instance, we certainly have $(\overline{C})^\circ = C$ and $D \subset \overline{D^\circ}$, but equality need not hold in the latter–we may lose components corresponding to the line at infinity.

Exercise 1.4.11. What on earth does this last line mean? Are the operations of homogenization and dehomogenization inverses to each other?

However, in projective space the coordinates X, Y, Z are completely symmetrical. This flexibility of homogenization with respect to one variable, applying a linear transformation of coordinates, and then dehomogenization with respect to another yields lots of birational transformations of curves.

Example 1.4.12. Consider the affine curve $x^3 + y^3 = \alpha$ from 1.2.B. The transformation of coordinates given by

$$(x_1, y_1) := \left(\frac{12\alpha}{x+y}, \frac{36\alpha(x-y)}{x+y}\right)$$

is a birational transformation onto the curve $y_1^2 = x_1^3 - 432\alpha^2$. Of course, this arises from homogenization first to $X^3 + Y^3 = \alpha Z^3$, applying the transformation 1.2.7, and then dehomogenization with respect to Z_1. (Make sure to convince yourself that this is a birational transformation!)

Here we used a linear change of coordinates in the middle. Let's study these a little more now.

Unimportant Remark 1.4.13. Sometimes there are equations "over $\mathbb{Z}$", which can then be reduced mod p for various primes p. For instance, we could look at the curve C given by $Y^2Z = X^3 + Z^3$, or equivalently $y^2 = x^3 + 1$, and count points over $\mathbb{F}_p$ for various p. One would like to say that the original curve is defined over something like $\mathbb{Z}$ (i.e., $C \subset \mathbb{P}^2_{\mathbb{Z}}$) and these $\mathbb{F}_p$-counts are the usual counts $C(\mathbb{F}_p)$ for the "extensions $\mathbb{F}_p/\mathbb{Z}$". This is subtle, but can be made to work using the modern language of schemes. Indeed, figuring out the right definition of $\mathbb{P}^2_{\mathbb{Z}}$ (it may not be what you think!) was an impetus for the development of modern scheme theory.

$$\begin{tikzcd}
C(\mathbb{Z}) \arrow[r, hook] \arrow[d, dashed, leftrightarrow] & C(\mathbb{Z}_p) \arrow[r] \arrow[d, dashed, leftrightarrow] & \overline{C}(\mathbb{F}_p) \arrow[r, hook] & \overline{C}(\overline{\mathbb{F}}_p) \\
C(\mathbb{Q}) \arrow[r, hook] \arrow[dr, hook] \arrow[ddr, hook] & C(\mathbb{Q}_p) \arrow[r, hook] & C(\overline{\mathbb{Q}}_p) \arrow[r, hook] & C(\mathbb{C}_p) \arrow[d, leftrightarrow, "\simeq"] \\
 & C(\mathbb{R}) \arrow[rr, hook] & & C(\mathbb{C}) \\
 & C(\mathbb{Q}[\sqrt[3]{2}]) \arrow[u, hook] \arrow[r, hook] & C(\mathbb{Q}[\sqrt[3]{2}, \zeta_3]) \arrow[r, hook] & C(\overline{\mathbb{Q}}) \arrow[u, hook]
\end{tikzcd}$$

Figure 1.4.14: A figure explaining the relationship between the K-points of a projective curve C defined over $\mathbb{Q}$ (and lifted to $\mathbb{Z}$) for various K. Some of the terminology and relationships will be explained later.

1.4.B Projective Changes of Coordinates

One advantage of working with projective spaces is the presence of lots of extra symmetries not visible in the affine plane. These come from linear changes of coordinates and look like

$$(X_1, Y_1, Z_1) = (aX + bY + cZ, dX + eY + fZ, gX + hY + iZ) \tag{1.4.15}$$

for $a, b, c, d, e, f, g, h, i \in K$. Clearly, not all choices of these "transformation coordinates" will work: you certainly can't take all of them to be zero. But, in fact the situation is a little stricter: you cannot take $(a, b, c, d, e, f, g, h, i) = (1, 0, 0, 1, 0, 0, 1, 0, 0)$ either. Similarly, but perhaps more subtly, you can't take $(a, b, c, d, e, f, g, h, i) = (2, 3, 5, -1, 2, -2, 0, 7, -1)$ either (why?). To see exactly what this constraint is, it is helpful to write this change of coordinates in the form

$$\begin{bmatrix} X_1 \\ Y_1 \\ Z_1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}. \tag{1.4.16}$$

The idea is that if this transformation is to work, then there better not be a triple of numbers X, Y, Z, not all zero, such that $X_1 = Y_1 = Z_1 = 0$. This amounts to saying that the matrix A as above "representing this transformation" has no kernel, or equivalently (by a little linear algebra) has nonzero determinant. Conversely, if this matrix has nonzero determinant, then we get a way to invert this transformation by applying A^{-1} on the left. This is therefore the correct condition for invertibility: $\det A \neq 0$.

This matrix representation is also very helpful when we compose changes of coordinates: this just amounts to multiplying the corresponding change-of-coordinate matrices. This would lead us to believe that the group (not abelian!) of changes of coordinates in the group of invertible matrices, $\mathrm{GL}_3(K)$. But it's a little more subtle than that. For instance, the change of coordinates $(X_1, Y_1, Z_1) = (\lambda X, \lambda Y, \lambda Z)$ for any $\lambda \in K^\times$ isn't *really* a change of coordinates: it's just the identity operation of $\mathbb{P}^2_K$. More generally, two matrices A and λA for $A \in \mathrm{GL}_3(K)$ and $\lambda \in K^\times$ represent the same change of coordinates. This tells us that the group of automorphisms is actually the quotient of $\mathrm{GL}_3(K)$ by its normal subgroup $K^\times$, which is called the *projective general linear group*

$$\mathrm{PGL}_3(K) := \mathrm{GL}_3(K)/K^\times.$$

Therefore, $\mathrm{PGL}_3(K)$ acts on $\mathbb{P}^2_K$ by linear changes of coordinates. (Similarly, $\mathrm{PGL}_{n+1}(K)$ acts on $\mathbb{P}^n_K$ for any $n \geq 1$. The action of $\mathrm{PGL}_2(K)$ on $\mathbb{P}^1_K$ is by *Möbius transformations.*)

Unimportant Remark 1.4.17. It can be shown (using heavy machinery that I will not get into here) that *all* (algebraic) automorphisms of $\mathbb{P}^2_K$ (over K) come from $\mathrm{PGL}_3(K)$: these are all the possible automorphisms you can have. Note that this is not true of $\mathbb{A}^2_K$: the map $(x, y) \mapsto (x, y - x^3)$ is a non-linear change of coordinates (i.e., automorphism) of $\mathbb{A}^2_K$.

Exercise 1.4.18. When does a change of coordinates preserve that line at infinity $Z = 0$? You will have to figure out what this means. For more along these lines, see 2.2.4.

The flexibility of these changes of coordinates, therefore, comes down to linear algebra. For instance, we can take any point to any other point by a linear change of coordinates. The same holds for any pair of points. The same is *not* true of any triple of points, since linear changes of coordinates preserve incidence relations: so you can't take a collinear triple of points to a non-collinear triple of points. But this is the only constraint: any collinear triple of points maps to any other, and any non-collinear triple maps to any other, by a linear change of coordinates. Finally, because projective changes of coordinates preserve incidence relations, this tells us that we can use them to map any line to any other. Similarly, any pair of lines goes to any other. Finally, not all triples of lines go to each other for the same reason as above: there is a difference between lines that "form a triangle" and coincident lines. Proving all of this amounts to a little linear algebra. You are invited to explore things of this sort in 2.1.13.

We have seen one example of change of coordinates in 1.2.B, and will see many examples of linear transformations throughout the course, so I don't feel terribly guilty giving too many examples now. Changes of coordinates are extremely useful, but lest you think those are all possible transformations, let me give you a warning:

Remark 1.4.19. Not every "change of coordinates" we will be interested in arises via a linear change of coordinates (on a projective closure). There are definitely other (birational) changes of coordinates we will be interested in. For a concrete example, see 2.1.12.