Symplectic Geometry and Hamiltonian Mechanics

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Contents

1 Introduction

The goal of this project is to expand on a paper by Sternberg on how to use symplectic geometry to couple a classical particle to a Yang-Mills field. He demonstrates that the symplectic procedure, for a general manifold M and gauge group G, generalizes the standard minimal coupling procedure for a particle in a classical electromagnetic field. The generalization allows for particles with charges that transform in nontrivial representations of the gauge group (which arise in nonabelian gauge theories). The idea is to use the connection form on the principal bundle to introduce a symplectic structure on certain associated bundles in a way that is automatically gauge invariant.

2 Symplectic Forms

In this section, we closely follow [1].

2.1 Fundamental Definitions

Definition 1. Let M be a smooth manifold. Then a *symplectic form* on M is a closed nondegenerate two form $\omega \in \Omega^2(M)$. A pair (M, ω) of a manifold and a symplectic form is called a *symplectic manifold*.

Example 1. Let $M = \mathbb{S}^2 \subset \mathbb{R}^3$ as the unit vectors, so that for $p \in \mathbb{S}^2$, the tangent space $T_p\mathbb{S}^2 \subset T_p\mathbb{R}^3 = \mathbb{R}^3$ is the set of vectors orthogonal to p. Then $\omega \in \Omega^2\mathbb{S}^2$ defined by $\omega_p(u, v) = \langle p, u \times v\rangle$ is symplectic. This can equivalently be written in cylindrical coordinates (θ, h) as $\omega = \mathrm{d}\theta \wedge \mathrm{d}h$.

Example 2. Let M be any manifold with local coordinates $q_1, \ldots, q_n$. Then any differential one form α in these local coordinates can be written as[1] $p^i \mathrm{d}q_i$. Then $(q_1, \ldots, q_n, p^1, \ldots, p^n)$ define local coordinates on the cotangent bundle T^*M, and we define the Liouville one-form $\alpha \in \Omega^1(T^*M)$ by $\alpha = p^i \mathrm{d}q_i$. (This can be checked to be well-defined independent of choice of coordinates.) Then $\omega = -\mathrm{d}\alpha = \mathrm{d}q_i \wedge \mathrm{d}p^i$ is a symplectic form on T^*M called the *canonical symplectic form*.

As a special case, take $M = \mathbb{R}^n$ with global coordinates q_i and p^i on $T^*\mathbb{R}^n \cong \mathbb{R}^{2n}$. Then the canonical symplectic form $\omega = \mathrm{d}q_i \wedge \mathrm{d}p^i$.

Why do physicists care about these objects? The answer lies in Theorem 1 below.

Definition 2. Let (M, ω) be a symplectic manifold and $H : M \to \mathbb{R}$ a smooth function, thought of as the Hamiltonian function. Then, by nondegeneracy of ω there is a unique vector field $X_H \in \Gamma(M, TM)$ such that the interior product $X_H \lrcorner\, \omega = \mathrm{d}H$, called the *Hamiltonian vector field associated to* H. Its flow is called the *Hamiltonian flow* generated by the Hamiltonian function H.

If ρ_t denotes the local flow of X_H, then it follows from

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t^*\omega = \rho_t^*\mathcal{L}_{X_H}\omega = \rho_t^*(\mathrm{d}(X_H \lrcorner\, \omega) + X_H \lrcorner\, (\mathrm{d}\omega)) = 0$$

that the flow of X_H preserves ω.

Example 3. If $(M, \omega) = (\mathbb{S}^2, \mathrm{d}\theta \wedge \mathrm{d}h)$, then taking $H = h : \mathbb{S}^2 \to \mathbb{R}$ to be the height function, we see that $X_H \lrcorner\, (\mathrm{d}\theta \wedge \mathrm{d}h) = \mathrm{d}h \Rightarrow X_H = \partial/\partial\theta$. The flow of X_H is $\rho_t(\theta, h) = (\theta + t, h)$, which clearly preserves the symplectic form ω and the Hamiltonain H.

Theorem 1. Consider $(M, \omega) = (\mathbb{R}^{2n}, \mathrm{d}q_i \wedge \mathrm{d}p^i)$, and let $H : M \to \mathbb{R}$ be any Hamiltonian. Then a curve $\gamma : I \to M$, $\gamma(t) = (q(t), p(t))$ is an integral curve of the vector field X_H iff Hamilton's equations of motion hold:

$$\dot{q}_i(t) = \left.\frac{\partial H}{\partial p^i}\right|_{\gamma(t)} \text{ and } \dot{p}^i(t) = -\left.\frac{\partial H}{\partial q_i}\right|_{\gamma(t)}.$$

[1] We're using the Einstein convention.

Proof. This amounts to calculting X_H. If $X_H = f_i \frac{\partial}{\partial q_i} + g^i \frac{\partial}{\partial p^i}$, then

$$(X_H \lrcorner \omega)(Y) = \omega(X_H, Y) = (\mathrm{d}q_i \wedge \mathrm{d}p^i)(X_H, Y) = \mathrm{d}q_i(X_H)\mathrm{d}p^i(Y) - \mathrm{d}q_i(Y)\mathrm{d}p^i(X_H) = f_i \mathrm{d}p^i(Y) - g^i \mathrm{d}q_i(Y).$$

Therefore,

$$f_i \mathrm{d}p^i - g^i \mathrm{d}q_i = X_H \lrcorner \omega = \mathrm{d}H = \frac{\partial H}{\partial p^i}\mathrm{d}p^i + \frac{\partial H}{\partial q_i}\mathrm{d}q_i.$$

Consequently,

$$f_i = \frac{\partial H}{\partial p^i} \text{ and } g^i = \frac{\partial H}{\partial q_i} \text{ so } X_H = \frac{\partial H}{\partial p^i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p^i}.$$

The claim follows because integral curves γ satisfy by definition $\dot{\gamma}(t) = X_H\big|_{\gamma(t)}$. ■

Therefore, the study of (integral curves of) Hamiltonian vector fields on symplectic manifolds is a vast generalization of the study of solutions to Hamiltonian's equations in Euclidean space.

2.2 Moment Maps

Definition 3. A smooth action ρ of a Lie group G on a symplectic manifold (M, ω) is *symplectic* if it acts by symplectomorphisms, i.e. for every $g \in G$ we have $\rho_g^* \omega = \omega$.

A symplectic action of $\mathbb{R}$ on a manifold gives us a one-parameter group of symplectomorphisms, whose derivative at $t = 0$ is a symplectic vector field.

Definition 4. A symplectic action of $\mathbb{R}$ on a manifold is *Hamiltonian* if its derivative at $t = 0$ is a Hamiltonian vector field.

Example 4. The action of $\mathbb{R}$ on $(\mathbb{S}^2, \mathrm{d}\theta \wedge \mathrm{d}h)$ given by $\rho_t(\theta, h) = (\theta + t, h)$ is a Hamiltonian action with its derivative at $t = 0$ being simply $\partial/\partial\theta = X_h$.

We can ask how to generalize this to arbitrary Lie groups, and here we provide the basic definitions. Recall that for a Lie group G, conjugating by $g \in G$ gives a diffeomorphism $\Psi_g : G \to G, h \mapsto ghg^{-1}$, whose derivative $\mathrm{Ad}(g) = T_e G \to T_e G$ is a linear automorphism of $T_e G = \mathfrak{g}$. Putting these together, we get a homomorphism $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$, which is a smooth representation of G called the *adjoint representation*. The *coadjoint representation* is the dual of this, i.e. the representation $\mathrm{Ad}^* : G \to \mathrm{GL}(\mathfrak{g}^*)$ with the property that for every $g \in G$, $\xi \in \mathfrak{g}^*$ and $v \in \mathfrak{g}$ we have $\langle \mathrm{Ad}_g^* \xi, v \rangle = \langle \xi, \mathrm{Ad}_{g^{-1}} v \rangle$.

Definition 5. Let ρ be a symplectic action of a Lie group G on a symplectic manifold M. Then the ρ is called *Hamiltonian* if there is a smooth map, *the moment map*, $\mu : M \to \mathfrak{g}^*$ with the following properties:

(a) Let $v \in \mathfrak{g}$, and let $\mu^v : M \to \mathbb{R}$ denote the map $\mu^v(p) = \langle v, \mu(p) \rangle$, where $\langle \cdot, \cdot \rangle : \mathfrak{g} \times \mathfrak{g}^* \to \mathbb{R}$ is the canonical pairing. If X_v is the vector field on M generated by the one-parameter subgroup $\{\exp(tv) : t \in \mathbb{R}\} \subset G$, then $\mathrm{d}\mu^v = X_v \lrcorner \omega$, i.e. μ^v is a Hamiltonian function for the vector field X_v.

(b) The map μ is equivariant w.r.t the action of G and the coadjoint representation, i.e. for all $g \in G$ we have $\mu \circ \rho_g = \mathrm{Ad}_g^* \circ \mu$.

The name comes from the fact that this generalizes the classical notions of momenta.

Example 5. Consider $(T^*\mathbb{R}^n, \mathrm{d}q_i \wedge \mathrm{d}p^i)$ and the left-action of $\mathbb{R}^n$ by translations: for $a \in \mathbb{R}^n$, we define $\rho_a(q, p) = (q + a, p)$. Consider the natural identifications $\mathfrak{g} = \mathbb{R}\langle \frac{\partial}{\partial a_i} \rangle$ and $\mathfrak{g}^* = \mathbb{R}\langle \mathrm{d}a_i \rangle$. Then for $v = v_i \frac{\partial}{\partial a_i}$, the vector field $X_v = v_i \frac{\partial}{\partial q_i}$ and the inner product $X_v \lrcorner \omega = v_i \mathrm{d}p^i = \mathrm{d}(v_i p^i)$, so that $\mu : M \to \mathfrak{g}^*$ must by given my $(q, p) \mapsto p^i \mathrm{d}a_i$, i.e. the classical momentum. Similarly, the SO_n action on $\mathbb{R}^n$ lifts to an action on $T^*\mathbb{R}^n$ by symplectomorphisms, which is Hamiltonian with respect to the angular momentum map $T^*\mathbb{R}^n \to \mathfrak{so}_n^*$, for example in dimension $n = 3$ given by the cross-product as usual.

When G is a connected Lie group, this can equivalently be described in terms of the *comoment map* $\mu^* : \mathfrak{g} \to \mathscr{C}^\infty(M)$ where the two conditions are rephrased as $\mu^*(v) = \mu^v$, and the fact that μ^* is a Lie algebra homomorphism, i.e. for $v, w \in \mathfrak{g}$ we have $\mu^*([v, w]) = \{\mu^* v, \mu^* w\}$, where the last is the *Poisson bracket* defined as $\{f, g\} := \omega(X_f, X_g)$. We now have all the technical tools set up.

3 The Yang-Mills Situation

A reference for the differential geometry background is [2]. A reference for Yang-Mills theory is [3].

Yang-Mills theory is a generalization of electromagnetism to gauge theories with nonabelian gauge groups. In order to describe Yang-Mills theory in the general setting, we will need to introduce the notion of a connection on a principal bundle.

Definition 6. Let G be a Lie group and M a smooth manifold. A *principal G-bundle* over M is a fiber bundle $\pi : E \to M$ with a smooth right action of G on E that is free, preserves the fibers of π, and acts transitively on the fibers.

Let $v \in \mathfrak{g}$ define a vector field X_v which associates to a point $p \in E$ the tangent vector at $t = 0$ of $p \exp(tv)$.

Definition 7. Let $\mathfrak{g} = T_e G$. A *connection* A on E is a $\mathfrak{g}$-valued one form that satisfies the following conditions: $A(X_v) = v$ for any $v \in \mathfrak{g}$, and for every $g \in G$ and $s \in TE$, $(R_g)_* A(s) = gA(s)g^{-1} = \mathrm{Ad}_g\, A(s)$, where R_g is the right action by g.

Example 6. For any Lie group G and manifold M, the product bundle $G \times M$ is a principal G-bundle with the action $g(h, x) = (hg^{-1}, x)$.

Example 7. The quotient map $\mathbb{S}^{2n+1} \to \mathbb{CP}^n$ defined by the action of $\mathrm{U}(1)$ by complex multiplication gives a principal $\mathrm{U}(1)$-bundle over $\mathbb{CP}^n$.

Definition 8. The *curvature* of a connection A is the $\mathfrak{g}$-valued two form $F_A = \mathrm{d}A + \frac{1}{2}[A \wedge A]$.

In Yang-Mills theory, a field configuration is a connection on the principal G-bundle over the spacetime manifold M, where G is the gauge group. The dynamics of the theory are defined by a Lagrangian involving the curvature of the connection.

Definition 9. The Yang-Mills Lagrangian density, for a connection A, is

$$L = -\frac{1}{2}\,\mathrm{Tr}\left(F_A \wedge \star F_A\right),$$

where $\star$ is the Hodge star operator.

To recast this in more standard physics language, Yang-Mills theory is a theory with fields A that take matrix values in the Lie algebra of the gauge group. The field strength is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + f^{abc} A^b_\mu A^c_\nu,$$

where μ and ν are spacetime indices, the a index runs over the generators of the Lie algebra, and the f^{abc} are the Lie algebra structure constants. The Lagrangian density is then

$$L = -\frac{1}{4} F^{\mu\nu a} F^a_{\mu\nu}.$$

4 Coupling to Charged Matter

In this section, we follow the argument in [4].

The Yang-Mills Lagrangian in the previous case did not include any charged matter coupled to the Yang-Mills fields. In this section we will describe a very general procedure to produce such a coupling. Let (F, ω) be a symplectic manifold and let G have a left action on F that is Hamiltonian, so that we have a moment map $\mu : F \to \mathfrak{g}^*$. Let $\pi : P \to M$ be a principal G-bundle for a compact Lie group G. The actions of G on P and F yield an action Q on $P \times F$, given by $Q_g(p, f) = (R_g p, L_g f) = (pg^{-1}, gf)$, for $g \in G$, $p \in P$, and $f \in F$. We will form the associated fiber bundle $P \times_G F \to M$ with fiber F by taking the quotient of $P \times F$ by the action of G.

Let A be a connection on P. By pairing A, which is a $\mathfrak{g}$-valued differential form on P, with the moment map, we get a real-valued differential form $\langle A, \mu\rangle$ on $P \times F$. Then

$$Q_g^* \langle A, \mu \rangle = \langle R_g^* A, L_g^* \mu \rangle = \langle \mathrm{Ad}_g A, \mathrm{Ad}_g^* \mu \rangle = \langle A, \mu \rangle.$$

The exterior derivative of this form is

$$\mathrm{d}\langle A, \mu \rangle = \langle \mathrm{d}A, \mu \rangle - \langle A, \mathrm{d}\mu \rangle,$$

where we interpret $\mathrm{d}\mu$ as a $\mathfrak{g}^*$-valued differential form. An element of the Lie algebra v and the action of G induces a vector field X_Q on $P \times F$, which can be written $X_Q = (X_P, X_F)$, where X_P and X_F are the vector fields induced by the G actions on P and F. Then

$$X_Q \lrcorner \mathrm{d}\langle A, \mu \rangle = \langle X_P \lrcorner \mathrm{d}A, \mu \rangle - \langle X_P \lrcorner A, \mathrm{d}\mu \rangle + \langle A, X_F \lrcorner \mathrm{d}\mu \rangle.$$

By the definition of a connection, we have $(R_g)^* A = \mathrm{Ad}_g A$. Differentiating this relation at the identity in G gives $D_{X_v} A = \mathrm{ad}_v A$ for $v \in \mathfrak{g}$. Using Cartan's formula, we have $D_{X_v} A = X_v \lrcorner \mathrm{d}A + \mathrm{d}(A(X_v))$. Since $A(X_v) = v$, we have $\mathrm{d}(A(X_v)) = 0$. Therefore $D_{X_v} A = X_v \lrcorner \mathrm{d}A = \mathrm{ad}_v A$. Using this relation in the previous calculation gives

$$X_Q \lrcorner \mathrm{d}\langle A, \mu \rangle = \langle \mathrm{ad}_v, \mu \rangle - \langle v, \mathrm{d}\mu \rangle + \langle A, \mathrm{ad}_v^* \mu \rangle = -\langle v, \mathrm{d}\mu \rangle = -X_F \lrcorner \omega,$$

where the last equality used the definition of the moment map. Since this is nonvanishing, $\mathrm{d}\langle A, \mu \rangle$ will not descend to a differential form on the quotient $P \times_G F$. However, if we pullback the symplectic form ω to $P \times F$ using the projection to F, then $X_Q \lrcorner \omega = X_F \lrcorner \omega$. Therefore $X_Q \lrcorner (\mathrm{d}\langle A, \mu \rangle + \omega) = -X_F \lrcorner \omega + X_F \lrcorner \omega = 0$. Furthermore,

$$Q_g^* \langle A, \mu \rangle = \langle R_g^* A, L_g^* \mu \rangle = \langle \mathrm{Ad}_g A, \mathrm{Ad}_g^* \mu \rangle = \langle A, \mu \rangle.$$

Also, $Q_g^* \omega = L_g^* \omega = \omega$ since the G action is symplectic. These two results imply that $\mathrm{d}\langle A, \mu \rangle + \omega$ descends to a form ω_A on $P \times_G F$. Because $\mathrm{d}\langle A, \mu \rangle + \omega$ is closed, so is ω_A.

Assume further that there is a symplectic form Ω on M. We can pullback this form to $P \times_G F$, and then $\Omega + \omega_A$ will be a closed two-form on $P \times_G F$. As long as this form is nondegenerate, this will define a symplectic structure on $P \times_G F$. In particular, when M is the cotangent bundle of another manifold with the canonical symplectic structure, then this form will be nondegenerate.

We will now show that this recovers the standard minimal coupling for electromagnetism. The typical procedure is to take the Hamiltonian H for the particle without any coupling to the field and substitute $p \mapsto p - edA$, where p is the four-momentum, A is the field potential, and e is the charge of the particle.

Let M be Minkowski space and $X = T^*M$ with the canonical symplectic form ω. Let $G = \mathrm{U}(1)$ and F a single point, with the moment map $\mu : F \to \mathfrak{g}^* = \mathbb{R}$ just picking out a point e. Let $P \to M$ be a principal U(1)-bundle and A a connection. Then we can form the pullback bundle $\tilde{P} \to T^*M = X$ and the connection lifts to a form on the pullback bundle. This form can be written in local coordinates as $A^i \mathrm{d}q_i$, where the q_i are local coordinates on X. Since F is just a point, $\tilde{P} \times_G F = X$. The symplectic form resulting from this procedure is $\omega + \mathrm{d}\langle A, \mu \rangle = \omega + \mathrm{d}(eA) = \omega + e\,\mathrm{d}A$.

Let $\phi : X \to X$ be the diffeomorphism defined by $(p, q) \mapsto (p + eA, q)$. Let H be a Hamiltonian function on X. The vector field ξ corresponding to the Hamiltonian $H(\phi^{-1}(p, q))$ (which corresponds to the replacement $p \mapsto p - eA$) generated by the standard symplectic form satisfies $\xi \lrcorner \omega = d(H \circ \phi^{-1}) = (\phi^{-1})^* \mathrm{d}H$. Notice that $\phi^*(\omega) = \omega + edA$, which is our modified symplectic form. We then have $\phi^*(\xi \lrcorner \omega) = (\phi^* \xi) \lrcorner (\omega + e\,\mathrm{d}A) = \mathrm{d}H$. Therefore $\phi^* \xi$ is the Hamiltonian vector field for H generated by the modified symplectic form. We see then that using modified symplectic form and the original Hamiltonian for the free particle gives the same dynamics as the original Hamiltonian with the minimal coupling substitution $p \mapsto p - eA$.

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