Moduli Spaces in AG & dim of Moduli of Curves /$\mathbb{C}$

May have been too ambitious.

Was going to make a rmk about first-order logic, but maybe too boring.

The Lefschetz Principle says that a thm in AG is true/$\mathbb{C}$ $\Leftrightarrow$ over any alg. closed field char 0

The first-order theory of AC fields of a fixed char (= 0) is Complete (by Łoś-Vaught test, which is itself an application of upward Lowen-Skolem). Related result by Lefschetz-Tarski which says that given a first order sentence φ in thry of fields, true/$\mathbb{C}$ $\Leftrightarrow$ all alg. closed fields of ch 0 $\Leftrightarrow$ alg. closed fields of ch $p \gg 0$).

Much of what I will say also works in positive char/ mixed characteristic/$\mathbb{Z}$

§1 Introduction.

$$\begin{tikzcd} X_b \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathfrak{X} \arrow[d, "\pi"] \\ b \arrow[r, hook] & B \end{tikzcd}$$

In any geometric Category, we want to study families of objects via invariants: discrete & cts.

$\{Z \subset X\}$.

Take all discrete invariants like deg, dim, Hilbert polynomial in flat family; remaining depends ctsly on parameters.

1. Def. Let $*$ be a kind of mathematical object. A **moduli space** for $*$ is an object $\mathcal{M}$ s.t.

special case.

(a) $\{\text{isom classes of } *\} \xleftrightarrow{\simeq} \{\text{pts of } \mathcal{M}\}$

(b) $\{\text{families } \mathfrak{X} \to B \text{ of } *\} \xleftrightarrow{\simeq} \{\text{morphisms } B \to \mathcal{M}\}$

At this pt, haven't said much. Could take $\mathcal{M}$ to be the set of isom classes.
Key pt that it is an object of the same category as $*$.
Universal family over $\mathcal{M}$ & every other formed by taking pullback

Fact. AG is the only kind of geometry in which (fd) moduli sp. exist.
Eg. If you fixed a manifold (smooth or topological - it doesn't matter) then the set of closed submanifolds of fixed dim does not make a fd mfld. But given a projective variety / $\mathbb{C}$, the set of closed subschemes of a fixed Hilbert poly. does form a projective variety / $\mathbb{C}$ (for suitable def. of variety...)

2. Examples.

(a) Fix fd VS $V/\mathbb{C}$. Then $*$ be $L \subset V$, i.e. one dim'l linear subsp.
$\mathcal{M} = \mathbb{P}V := (V - \{0\}) / \mathbb{C}^\times \leftarrow$ Quotients by gp actions show up early on in the theory of moduli spaces

Similarly, $Gr(d, V)$, $Fl(\underline{d}, V)$, etc.

(b) Fix $\mathbb{P}^n$ & $d \in \mathbb{Z}_{\geq 0}$. Then $*$ be hypersurfaces $X \subset \mathbb{P}^n$ of deg d.

$$\mathcal{M} = (\mathbb{C}[x_0, \dots, x_n]_d - \{0\}) / \mathbb{C}^\times = \mathbb{P} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong \mathbb{P}^{\overbrace{\binom{n+d}{d} - 1}^{N}}$$

& $\exists$ open $U \subset \mathbb{P}^N$ consisting of smooth hypersurfaces & U / PGL_{n+1}.

Rmks (i) This $\mathcal{M}$ is an interesting compactification of U - the complement parametrizes nonreduced or reducible hyp. Not the only possible compact.

(ii) There is a universal family over $\mathcal{M}$ called universal hyp. of deg d s.t. every other flat family given by pulling back. This is an example of a Hilbert scheme, which are fascinating & imp. moduli sp.
(iii) Embedded hypersurfaces & dependent on coordinates. To do coordinate-independently, mod out by PGL_{n+1}.

3. **Facts**

(i) If X is an irred alg. variety $/\mathbb{C}$ & $U \subset X$ nonempty open, then U is dense & $\dim U = \dim X$.

This lets us restrict to open subsets of „generic behavior".

(ii) Let $f : X^m \to Y^n$ be a morphism of smooth varieties. If Y irreducible $\forall y \in Y$, $f^{-1}(y)$ irreducible of same dim d, then X is irreducible & $m = d + n$. Eg. $\dim X/G = \dim X - \dim G$.

Glossing over technicalities.

4. **Examples**.

(a) Given a variety X and $d \in \mathbb{Z}_{\geq 0}$, consider

$$\mathrm{Sym}^d X := \{ \{p_1, \ldots, p_d\} : p_1, \ldots, p_d \in X \}.$$

$$\pi : X^d - \Delta \longrightarrow \mathrm{Sym}^d X \quad \& \text{ so } \dim \mathrm{Sym}^d X = d \cdot \dim X$$

(b) If a general hyp. surface of deg $d \subset \mathbb{P}^n$ has no aut., then

$$U \longrightarrow U/PGL_{n+1} \;\therefore\; \dim = N - (n+1)^2 - 1.$$

Eg. $n = 2$, $d = 4$ gives $14 - 8 = 6 = 3 \cdot 3 - 3$ ← plane quartics $g = 3$.

$n = 3$, $d = 4$ gives $34 - 15 = 19$ ← moduli of K3 surfaces

§2 Moduli of Curves

1. Def. A **Curve** is a smooth irreducible projective variety$_{/\mathbb{C}}$ of dim 1

These are classified first by genus.

For each $g, n \in \mathbb{Z}_{\geq 0}$,

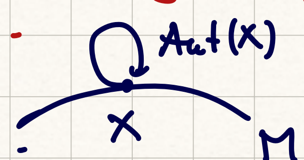
$$M_{g,n} := \{ (X; p_1, \ldots, p_n) : X \text{ of genus } g,\ p_1, \ldots, p_n \in X \} / \cong.$$

Theorem that exists as quasiprojective variety for $g, \underline{n}$.

$\exists$ various compactifications eg. Deligne-Mumford $\overline{M}_{g,n}$ „stable Curves"

Modern perspective says: study these as stacks/orbifolds, amts to remembering automorphisms. Analogous to counting # thry & gp thry Count by wting by automorphisms & / Aut(X) gps.

analogy w/ Bass-Serre trees



2. Thm. (Riemann, 1857) $\dim M_{g,n} = 3g-3+n$ for $g \geq 2$

„dim" = dim M − dim Aut.

Riemann introduced moduli: Parameter Count. 50 yrs before defn of manifold, 100 before "scheme", shows how classical these objects are.

„Theorie der Abel'schen Funktionen"

Note: $M_{g,n} \xrightarrow{\pi} M$ has fibers C^n $\therefore$ $\underbrace{\dim M_{g,n} \overset{"}{=} \dim M_g + n}_{\text{subtlety about automorphisms, get to in a second}}$

$\boxed{\text{Pf 1}}$

$g=0.$ $X \cong \mathbb{P}^1$ & $M_0 = *$.

$\mathrm{Aut}(\mathbb{P}^1) = PGL_2^{\,3}$ & so $\mathcal{M}_0 = [*/PGL_2]^{-3}$ $= 3(0)-3$

In algebraic geometry, dim = # variables − # equations. Negative dim says that more equations than variables & is an effective way of keeping track of how many more, so when you add more variables, you can get the correct dim ct.

For $n \geq 3$,

$$(\mathbb{P}^1 \setminus \{0,1,\infty\})^{n-3} \setminus \Delta \hookrightarrow M_{0,n}^{\,n-3}$$

$$p_4,\dots,p_n \longmapsto [\mathbb{P}^1; 0,1,\infty,p_4,\dots,p_n].$$

$g=1.$ $X \cong$ Smooth Cubic $\subset \mathbb{P}^2$, classified by j $\therefore$ $M_1 \cong \mathbb{A}^1_{\mathbb{C}}$.

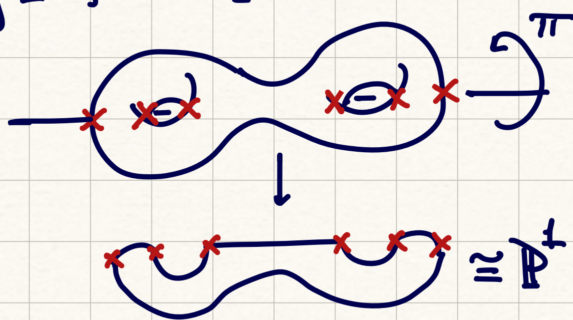
These are the elliptic curves. $\mathbb{H}/PSL_2\mathbb{Z} \xrightarrow{j} \mathbb{C}$ hol. function, mod form et

$\mathrm{Aut}(X)$ has dim 1 $\therefore$ $\dim \mathcal{M}_1 = 0 = 3(1)-3$.

Abelian variety; fixing an $O \in X$, get translation morphisms. Aut which fix O are finite.

$g \geq 2$: $\mathrm{Aut}(X) \leq 84(g-1)$ $\therefore$ $\dim \mathcal{M}_g = \dim M_g$.

$g=2$ X is hyperelliptic, i.e. $\exists$ deg 2 $f: X \to \mathbb{P}^1$ branched cover of $\mathbb{P}^1$ branched at 6 pts.



$$\mathcal{M}_2^{\,3} \rightleftarrows M_{0,6}^{\,3}$$

Essentially unique / PGL_2

$g \geq 2$. Study via maps to $\mathbb{P}^1$. (1891)

Fix $d \gg 0$. $\mathrm{Hur}_{d,g} = \{(X,f) : X \in M_g,\ f : X \xrightarrow{d} \mathbb{P}^1\}/\cong$.

By RH,

$$2g-2 = d\cdot(2(0)-2) + R \Rightarrow R = \#\,\text{branchpts} = 2d+2g-2.$$

Simply branched $U \subset \mathrm{Hur}_{d,g}$.

$$\begin{tikzcd}
\mathrm{Hur}_{d,X}^{\,2d-g+1} \arrow[r] \arrow[d] & \mathrm{Hur}_{d,g}^{\,2d+2g-2} \arrow[dl] \arrow[dr, "\text{dominant, finite fibers}"] & \\
{[X]} \arrow[r, mapsto] & M_g^{\,3g-3} & \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1
\end{tikzcd}$$

$$\mathrm{Hur}_{d,C}^{\,2d-g+1} = \{ f : X \xrightarrow{d} \mathbb{P}^1 \} = \left\{ (L,\sigma,\tau) : \begin{array}{l} L \in \mathrm{Pic}^d(X), \\ \sigma,\tau \in H^0(X,L) \end{array} \right\} \Big/ (\sigma,\tau)\sim(\lambda\sigma,\lambda\tau)$$

$$\Big\downarrow\ 2\underbrace{h^0(X,L)}_{(d-g+1)}-1 = 2d-2g+1$$

$$d-g = h^0(X,L)-1$$

$$\left\{ (L,\sigma) : \begin{array}{l} L \in \mathrm{Pic}^d X, \\ \sigma \in H^0(X,L) \end{array} \right\} \Big/ \text{scaling} \cong \mathrm{Sym}^d C^{\,d} \twoheadrightarrow \mathrm{Pic}^d(X)^{\,g} \to \text{Jacobian variety}$$

$$\{p_1,\dots,p_d\} \longmapsto \mathcal{O}_X(p_1+\dots+p_d). \qquad \blacksquare$$

Rmk. $\mathcal{H}_{d,g} = \mathrm{Hur}_{d,g}/PGL_2$ has $\dim = 2d+2g-5$ & have branch morphism $\mathcal{H}_{d,g} \longrightarrow \mathcal{M}_{0,2d+2g-2}$.

$\boxed{\text{Pf 2}}$

1. If $\mathcal{M}$ a moduli space, then for $X \in \mathcal{M}$, $T_X\mathcal{M}$ classifies 1^{st} order deformations of X. $\longrightarrow$ Hilbert/Fano Schem

$$\begin{tikzcd} \operatorname{Spec} \mathbb{C}[x]/(x^2) \arrow[dr] & \\ \operatorname{Spec} \mathbb{C} \arrow[u, hook] \arrow[r, "X"] & \mathcal{M} \end{tikzcd}$$

Flat Families / $\mathbb{C}[x]/(x^2)$ with Central fiber $\cong X$.

2. If X is a smooth variety$/\mathbb{C}$, 1^{st} order def of X classified by $H^1(X, TX)$.

$$\kappa : T_X M \xrightarrow{\sim} H^1(X, TX).$$

$\therefore \dim M = h^1(X, TX)$.

If X is a curve of genus g, then

$$\begin{aligned} \dim M_g &= h^1(X, TX) \\ &= h^1(X, K_X^\vee) \\ &= h^0(X, K_X^{\otimes 2}) && \text{Serre duality} \\ &= \deg K_X^{\otimes 2} - g + 1 + h^0(X, TX) && \text{Riemann-Roch} \\ &= 2 \deg K_X - g + 1 + \cdots && \text{Additivity of deg} \\ &= 2(-\chi_{top}(X)) - g + 1 + \cdots && \text{Def of } \chi \text{ / Gauß-B.} \\ &= 2(2g-2) - g + 1 + \cdots \\ &= 3g - 3. \end{aligned}$$

$$+ h^0(X, TX) = \dim \operatorname{Aut}(X) = \begin{cases} 3 & g=0 \\ 1 & g=1 \\ 0 & g \geq 2. \end{cases}$$

$\boxed{\text{Pf 3}}$ Hyperbolic Surfaces & Teichmüller Theory

Trichotomy in uniformization of cpt RS : $g=0$ round, $g=1$ flat, $g \geq 2$ hyperbolic

For $g \geq 2$, fix Σ_g smooth oriented closed surface of genus g, our top. model
For CRS X of genus g, a marking is a diffeo $f : \Sigma_g \to X$.

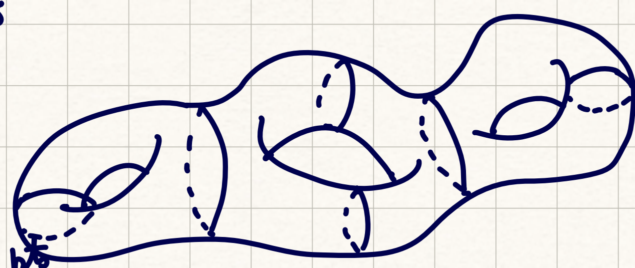
$$\mathcal{T}_g := \{(X, f) : X \text{ cRS of gen } g,\ f : \Sigma_g \to X \text{ marking}\} / \sim$$

where $(X, f) \sim (Y, g)$ if $\exists$ $\mathbb{C}$-iso $\alpha : X \to Y$ s.t.
$$\begin{tikzcd} & X \arrow[d, "\alpha"] \\ \Sigma_g \arrow[ur, "f"] \arrow[dr, "g"'] \arrow[r, phantom, "\circlearrowright"] & Y \end{tikzcd}$$
up to isotopy

Then map $\pi : \mathcal{T}_g \longrightarrow \mathcal{M}_g$.

Fact 1. $\mathcal{T}_g$ can be naturally made into a $\mathcal{C}^\infty$ mfld (symplectic & $\mathbb{C}$-mfld) st π is $\mathcal{C}^\infty$ (holom.).
Fact 2. $\exists$ discrete gp $MCG^+(\Sigma_g) \curvearrowright \mathcal{T}_g$ prop. disct s.t.
$$\begin{tikzcd} \mathcal{T}_g \arrow[rr, "\pi"] \arrow[dr] & & \mathcal{M}_g \\ & \mathcal{T}_g / MCG^+(\Sigma_g) \arrow[ur] & \end{tikzcd}$$
$\therefore$ Covering space.
Fact 3. $\mathcal{T}_g \cong \mathbb{R}^{2N}$ for some N, $N = \dim_{\mathbb{C}} \mathcal{T}_g$.
$\therefore$ Contractible, $\mathcal{T}_g$ is universal cover of $\mathcal{M}_g$ w/ $\pi_1 = MCG^+(\Sigma_g)$.
Fact 4. Can give coordinates on $\mathcal{T}_g$ via pairs of pants decomposition.
„Fenchel-Nielsen Coordinates"

$$(l_i, \tau_i)_{i=1}^N : \mathcal{T}_g \xrightarrow[\mathcal{C}^\infty]{\cong} \mathbb{R}^{2N}$$



and $N = \#$ geodesics in pair of pts decomposition $= 3g-3$.

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