Mathematical Ghostbusters: Despookifying Spectral Sequences

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Abstract

This is an expanded version of a collection of lecture notes based on a mini talk series aimed at counselors I gave at Ross/Ohio 2024. The goal of this set of notes is to develop the basic theory of spectral sequences in a complete yet digestible manner, followed by a plethora of illustrating examples drawn from a variety of subfields of math.

Contents

1 Preliminary Remarks

Spectral sequences are often described as "messy" [1, p. 442], "tedious, but elementary" [2, p. 5], and even " terrifying, evil, and dangerous [like specters]" [3].[1] The goal of this set of lecture notes is to argue otherwise.

We develop the basic theory of spectral sequences using filtered differential modules. The same arguments can more or less be carried out in any abelian category (see Remark 2.8 for an example of where we need to be a little careful), but we stick to modules over a(n) (unspecified) fixed ring for simplicity. Although sometimes useful (as in the construction of the Bockstein spectral sequence), exact couples do not show up in this version of the notes, and neither does the product structure on spectral sequences.[2] Finally, the more advanced reader knows that the cleanest, although perhaps pedagogically not the most effective, way to understand spectral sequences is via the language of derived categories, which we leave to other sources.

Conventions

- Throughout the text, we fix a base ring; a *module* is a left module over this base ring.
- The class of an element $x \in C$ in a quotient module C/C' of C is denoted by $[x]_{C/C'}$.
- We follow the cohomological notational convention for spectral sequences; of course, the homological convention can be obtained by switching subscripts and superscripts, and applying the automorphism $n \mapsto -n$ to every indexing $\mathbb{Z}$.
- When dealing with double complexes, we will use the convention where the vertical and horizontal differentials anticommute, i.e. that $\partial_v \partial_h + \partial_h \partial_v = 0$. Some authors prefer the convention that they commute, i.e. that $\partial_v \partial_h = \partial_h \partial_v$ instead. These approaches are clearly equivalent under the transformation $\partial_v^{p,q} \mapsto (-1)^q \partial_v^{p,q}$.

[1] This sentence is also an excuse to list other standard references on the theory of spectral sequences.
[2] Perhaps they will be present in a future version.

2 Introduction

2.1 Differential Modules and Spectral Sequences

Definition 2.1.

(a) A **differential module** is a pair (C, ∂), where C is a module and $\partial : C \to C$ is an endomorphism such that $\partial^2 = 0$.
(b) A **morphism** $f : (C, \partial) \to (C', \partial')$ **of differential modules** is a morphism $f : C \to C'$ that commutes with the differentials, i.e., that satisfies $\partial' f = f \partial$.

The condition $\partial^2 = 0$ is equivalent to saying that the submodule $B := \operatorname{im} \partial$ of **coboundaries** is contained in the submodule $Z := \ker \partial$ of **cocycles**, and we define the **cohomology** of (C, ∂) to be

$$\mathrm{H}C = \frac{Z}{B} = \frac{\ker \partial}{\operatorname{im} \partial}.$$

Taking cohomology is functorial: a morphism $f : (C, \partial) \to (C', \partial')$ of differential modules induces a map

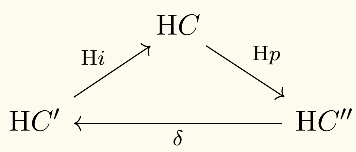
$$\mathrm{H}f : \mathrm{H}C \to \mathrm{H}C'.$$

Remark 2.2. Sometimes we will use the notation $B(\partial) \subset Z(\partial)$ and $\mathrm{H}(C, \partial)$ if we want to emphasize ∂. Having said this, we will often drop the ∂ and call C itself the differential module, and also drop all notational embellishment (e.g. primes) on ∂, using ∂ to refer to the differential of any differential module. Similar notational sleights-of-hand will be made without further comment.

Theorem 2.3. A short exact sequence of differential modules

$$0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$$

induces an **exact triangle in cohomology**, i.e., there is a morphism $\delta : \mathrm{H}C'' \to \mathrm{H}C'$ that makes the triangle



exact.

This is a standard diagram chasing argument, and the only such argument we will give (and need).

Proof. Given a class $[c] \in \mathrm{H}C''$, pick a representative $c \in C''$ of $[c]$ and a $b \in C$ with $p(b) = c$. Then $p(\partial b) = \partial p(b) = \partial c = 0$, so that $\partial b \in \ker p = \operatorname{im} i$, and hence there is an $a \in C'$ such that $i(a) = \partial b$. Then $i(\partial a) = \partial i(a) = \partial^2 b = 0$, whence a is a cocycle and hence $[a] \in \mathrm{H}C'$. The map $\delta : \mathrm{H}C'' \to \mathrm{H}C'$ takes $[c] \mapsto [a]$. It is standard to check that this is well-defined irrespective of the choices made above, and makes the above triangle exact. ∎

Remark 2.4. Of course, the data of a short exact sequence $0 \to C' \to C \to C'' \to 0$ amounts to a submodule $C' \subset C$ and an identification $C/C' \to C''$ of the quotient module with C''. With this perspective, the map δ in the above theorem, called the **connecting homomorphism**, is given by the recipe "pick representative in C and take ∂"–this will be very helpful to keep in mind in what follows.

Now we are ready to define spectral sequences.

Definition 2.5. A (cohomological) spectral sequence is a sequence $E = (E_r, \partial_r)_{r\geq 0}$ of differential modules and given isomorphisms $\mathrm{H}(E_r, \partial_r) \xrightarrow{\sim} E_{r+1}$ for each $r \geq 0$.

We leave it to the reader to formulate the notion of a morphism of spectral sequences. The differential modules in the sequence E are often called the pages of the spectral sequence, for reasons that should soon be clear. The essential data of a spectral sequence is contained in the E_0 page and the differentials ∂_r for $r \geq 0$; we will often use the given isomorphisms to implicitly identify $\mathrm{H}(E_r, \partial_r)$ with E_{r+1} for each $r \geq 0$.

Given a spectral sequence E, each page E_r for $r \geq 0$ is a subquotient[3] of E_0. This is made precise by

Proposition 2.6. If E is a spectral sequence, then there is a chain of submodules

$$0 = B_0 \subset B_1 \subset B_2 \subset \cdots \subset Z_2 \subset Z_1 \subset Z_0 = E_0$$

and, for each $r \geq 0$, a surjection $\pi_r : Z_r \to E_r$ with kernel B_r, giving us an identification

$$Z_r/B_r \xrightarrow{\sim} E_r.$$

Proof. Take $B_0 = 0$ and $Z_0 = E_0$ with $\pi_0 = \mathrm{id}_{E_0}$. Inductively, for each $r \geq 0$, let

$$B_{r+1} := \pi_r^{-1} B(\partial_r) \subset \pi_r^{-1} Z(\partial_r) =: Z_{r+1} \text{ and } \pi_{r+1} := [\cdot]_{E_{r+1}} \circ \pi_r|_{Z_{r+1}},$$

where $[\cdot]_{E_{r+1}} : Z(\partial_r) \to E_{r+1}$ takes classes modulo $B(\partial_r)$, using the given identification $\mathrm{H}(E_r, \partial_r) \xrightarrow{\sim} E_{r+1}$. ■

Proposition 2.6 also works in the "other direction": to give a spectral sequence E, it also suffices to give a module E_0 with a chain of submodules as in the proposition, and to give the differentials $\partial_r : Z_r/B_r \to Z_r/B_r$ for each $r \geq 0$.

Definition 2.7. Given a spectral sequence E and submodules B_r, Z_r for $r \geq 0$ as in Proposition 2.6, we define

$$B_\infty := \bigcup_{r\geq 0} B_r \subset \bigcap_{r \geq 0} Z_r =: Z_\infty \text{ and } E_\infty := Z_\infty / B_\infty.$$

It is customary to call the module E_∞, the "infinite-th" (or the "infinith," or the "infinity") page of the spectral sequence E, and to say that E converges to E_∞.

Remark 2.8. In a general abelian category, the (co)limits defining Z_∞ and B_∞ may not exist, but we don't need to worry too much about that because: these certainly exist in the category of modules over a fixed ring, and, in practice, most spectral sequences we will work with degenerate at some finite stage.

[3] The sub- of quotient-, which is also the quotient- of a sub-.

Remark 2.9. For each r with $1 \leq r \leq \infty$, we have

$$Z_r = \{x \in E_0 : (\forall\, j : 1 \leq j < r) \text{ we have } \partial_j \left([x]_{E_j}\right) = 0\}, \text{ and}$$
$$B_r = \{x \in E_0 : (\forall j : 1 \leq j < r)(\exists\, y_j \in E_j) \text{ such that } [x]_{E_j} = \partial_j y_j\}.$$

The conditions defining these sets are somewhat recursive: for each $j \geq 0$, we need to have $\partial_j([x]_{E_j}) = 0$ in order for $[x]_{E_{j+1}}$ to be defined. For this reason, for each $0 \leq r \leq \infty$, I like to call Z_r the submodule of r-deep cocycles and B_r the submodule of r-deep coboundaries; this is, of course, not standard terminology.

Often the differential modules we want to work with will either be graded or filtered. Let's talk about gradings and filtrations now.

2.2 Gradings

Fix an abelian group G.

Definition 2.10.

(a) A G-graded module is a module C along with submodules $C^g \hookrightarrow C$ indexed by $g \in G$ such that C is the internal direct sum of the C^g's, i.e., the natural map $\bigoplus_{g\in G} C^g \to C$ is an isomorphism.

(b) A morphism $f : C \to C'$ of G-graded modules is a pair (f, d), where $f : C \to C'$ is a morphism of modules and $d \in G$ such that for all $g \in G$, we have $f(C^g) \subset (C')^{g+d}$. The element $d \in G$ is called the degree of f is denoted by $\deg f$.

Now we would like to combine the structures from Definitions 2.1 and 2.10.

Definition 2.11.

(a) A G-graded differential module is a pair (C, ∂), where C is a G-graded module and $\partial : C \to C$ is an endomorphism of G-graded modules such that $\partial^2 = 0$.

(b) A morphism $f : (C, \partial) \to (C', \partial')$ of G-graded differential modules is a morphism $f : C \to C'$ that is both a morphism of differential modules and of G-graded modules.

The degree $\deg \partial \in G$ of ∂ is called the degree of the G-graded differential module (C, ∂). If $f : (C, \partial) \to (C', \partial')$ is a morphism of G-graded differential modules, then the relation $\partial' f = f\partial$ implies $\deg \partial' = \deg \partial$ in all but the most degenerate of cases, and we will assume this equality of degrees to be the case whenever we speak of such morphisms.

Remark 2.12. If C is a G-graded differential module, then Z, B and $\mathrm{H}C$ are all G-graded via

$$B^g := B \cap C^g \subset Z \cap C^g =: Z^g \text{ and } \mathrm{H}^g C = Z^g/B^g \hookrightarrow \mathrm{H}C$$

for $g \in G$. If $f : C \to C'$ is a morphism of such modules, then the induced morphism in cohomology $\mathrm{H}f : \mathrm{H}C \to \mathrm{H}C'$ is a morphism of G-graded modules as well, with $\deg \mathrm{H}f = \deg f$. Finally, if $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ is a short exact sequence of such modules, then the connecting homomorphism $\delta : \mathrm{H}C'' \to \mathrm{H}C'$ is graded of degree

$$\deg \delta = \deg \partial - \deg i - \deg p,$$

as is clear from the proof of Theorem 2.3 (see also Remark 2.4). This amounts to saying that if you "go around" the exact triangle once, then you change the degree by exactly $\deg \partial$.

Example 2.13. A cochain complex is a differential $\mathbb{Z}$-graded module C of degree $\deg \partial = 1$; such an object is often denoted as $(C^\bullet, \partial)$. If $0 \to C' \xrightarrow{i} C \xrightarrow{p} C'' \to 0$ is a short exact sequence of cochain complexes with $\deg i = \deg p = 0$, then the exact triangle in cohomology from Theorem 2.3 "unfolds" to give us the usual long exact cohomology sequence

$$\cdots \to \mathrm{H}^q C' \xrightarrow{\mathrm{H}i} \mathrm{H}^q C \xrightarrow{\mathrm{H}p} \mathrm{H}^q C'' \xrightarrow{\delta} \mathrm{H}^{q+1} C' \to \cdots$$

The notion of a G-grading can also be defined for spectral sequences.

Definition 2.14. A G-graded spectral sequence is a spectral sequence $E = (E_r, \partial_r)_{r\geq 0}$ such that each page (E_r, ∂_r) is a G-graded differential module and the given isomorphisms $\mathrm{H}E_r \to E_{r+1}$ are morphisms of G-graded modules.

If E is a G-graded spectral sequence, setting

$$B_r^g := B_r \cap E_0^g \subset Z_r \cap E_0^g =: Z_r^g$$

for all $0 \leq r \leq \infty$ and $g \in G$ gives G-gradings on the modules B_r and Z_r such that the morphisms $\pi_r : Z_r \to E_r$ are morphisms of graded G-modules for all $r \geq 0$. We give E_∞ a G-grading by defining $E_\infty^g := [Z_\infty^g]_{E_\infty}$ for all $g \in G$, which makes

$$0 \to B_\infty \to Z_\infty \to E_\infty \to 0$$

a short exact sequence of G-graded modules. Finally, we invite the reader to define morphisms of G-graded spectral sequences, and check that they induces morphisms of G-graded modules on all the B_r, Z_r and E_∞.

2.3 Filtrations

Definition 2.15.

(a) A (descending or cohomological) filtration F on a module C is a sequence $(F^pC)_{p\in\mathbb{Z}}$ of submodules $F^pC \subset C$ such that for all $p \in \mathbb{Z}$ we have $F^pC \supset F^{p+1}C$. The filtration is usually denoted by writing

$$F^{-\infty}C := C \supset \cdots \supset F^pC \supset F^{p+1}C \supset \cdots .$$

A filtered module is a pair (C, F), where C is a module and F a filtration on C.

(b) A morphism of filtered modules $f : C \to C'$ is a morphism f respecting the filtrations, i.e., such that for all $p \in \mathbb{Z}$ we have $f(F^pC) \subset F^pC'$.

More generally, we can also define filtrations indexed by totally ordered abelian groups, or morphisms that alter the filtering degree, but we will not need these notions in what follows. Given a filtered module C, the graded module $\operatorname{Gr} C$ associated to C is the $\mathbb{Z}$-graded module defined by

$$\operatorname{Gr} C = \bigoplus_{p\in\mathbb{Z}} \operatorname{Gr}^p C \text{ where } \operatorname{Gr}^p C = F^pC/F^{p+1}C \text{ for } p \in \mathbb{Z}.$$

A morphism $F : C \to C'$ of filtered modules induces a morphism of $\operatorname{Gr} f : \operatorname{Gr} C \to \operatorname{Gr} C'$ of the associated graded modules.

Finally, we can combine Definitions 2.1 and 2.15 to arrive at

Definition 2.16.

(a) A filtered differential module is a pair (C, ∂), where C is a filtered module and $\partial : C \to C$ is an endomorphism of filtered modules such that $\partial^2 = 0$.

(b) A morphism $f : (C, \partial) \to (C', \partial')$ of filtered differential modules is a morphism $f : C \to C'$ that is both a morphism of differential modules and of filtered modules.

Similarly, one can define filtered spectral sequences, although we will not need this notion either.

Remark 2.17. If C is a filtered differential module, then $(F^pC, \partial|_{F^pC})$ is a differential module for each $p \in \mathbb{Z}$, and and the inclusion $F^pC \hookrightarrow F^qC$ is a morphism of differential modules for each $p \geq q \geq -\infty$, yielding maps $\mathrm{H}F^pC \to \mathrm{H}F^qC$. From this, we get two things:

(a) The associated graded module $\operatorname{Gr} C$ acquires the structure of a $\mathbb{Z}$-graded differential module of degree 0 such that for all $p \in \mathbb{Z}$ we have

$$\mathrm{H}^p \operatorname{Gr} C = \mathrm{H}(F^pC/F^{p+1}C).$$

(b) We get filtrations on B, Z and $\mathrm{H}C$ via

$$F^pB := B \cap F^pC \subset Z \cap F^pC =: F^pZ \text{ and } F^p\mathrm{H}C := \operatorname{im}(\mathrm{H}F^pC \to \mathrm{H}C)$$

for each $p \in \mathbb{Z}$.[4] The graded module associated to $\mathrm{H}C$ is then given by

$$\operatorname{Gr}^p \mathrm{H}C := \frac{\operatorname{im}(\mathrm{H}F^pC \to \mathrm{H}C)}{\operatorname{im}(\mathrm{H}F^{p+1}C \to \mathrm{H}C)} \cong \frac{F^pZ}{F^pB + F^{p+1}Z}$$

for $p \in \mathbb{Z}$.

[4] Note that the maps $\mathrm{H}F^pC \to \mathrm{H}C$ need not be injective in general, and also that $F^pZ = Z(\partial|_{F^pC})$ and $F^pB \supset B(\partial|_{F^pC})$, but the last containment can be proper in general.

We may now ask: given a filtered differential module C, what is the relationship between the $\mathbb{Z}$-graded modules $\mathrm{H}^\bullet \operatorname{Gr} C$ and $\operatorname{Gr}^\bullet \mathrm{H}C$? In general, there need not be a map between them in any direction; however, it seems implausible that they are not related altogether. Let's first examine a special case.

Example 2.18. A short exact sequence $0 \to C' \to C \to C'' \to 0$ of differential modules induces a two-step filtration

$$C \supset C' \supset 0$$

on C via the identifications mentioned in Remark 2.4 (i.e., $F^pC = C$ for $p \leq 0$, and $F^1C = C'$, and $F^pC = 0$ for $p \geq 2$). In this case, we have in degrees 0 and 1 when read from left to right (check!) that

$$\begin{aligned} \operatorname{Gr} C &= C'' && \oplus C', \\ \mathrm{H} \operatorname{Gr} C &= \mathrm{H}C'' && \oplus \mathrm{H}C', \text{ and} \\ \operatorname{Gr} \mathrm{H}C &= \frac{\mathrm{H}C}{\operatorname{Im}(\mathrm{H}C' \to \mathrm{H}C)} && \oplus \operatorname{Im}(\mathrm{H}C' \to \mathrm{H}C). \end{aligned}$$

From this chart, the relationship between $\mathrm{H} \operatorname{Gr} C$ and $\operatorname{Gr} \mathrm{H}C$ becomes clear: Theorem 2.3 gives us a map $\delta : \mathrm{H}^0 \operatorname{Gr} C \to \mathrm{H}^1 \operatorname{Gr} C$ making $(\mathrm{H}^\bullet \operatorname{Gr} C, \delta)$ a cochain complex with cohomology $\operatorname{Gr} \mathrm{H}C$.

In general, at least when the filtration on C is bounded and exhaustive (to be defined soon), the correct answer to the above question is that there is a $\mathbb{Z}$-graded spectral sequence E such that

(a) for each $r \geq 0$, we have $\deg \partial_r = r$ (so, e.g., the E_1 page is cochain complex),
(b) for $r = 0$ we have $E_0 = \operatorname{Gr} C$ with induced differentials, so that $E_1 = \mathrm{H} \operatorname{Gr} C$, and
(c) for $r = \infty$, we have an isomorphism $E_\infty \cong \operatorname{Gr} \mathrm{H}C$.

In Example 2.18, this spectral sequence collapsed at the E_2 page, i.e., we had $E_2 \cong E_\infty$, and the isomorphism $E_\infty \cong \operatorname{Gr} \mathrm{H}C$ was the content of Theorem 2.3. Our next goal is to prove this general result; as we shall see, it is the key to understanding spectral sequences.

2.4 Product Structures: (Filtered) Differential Graded Algebras*

3 Fundamental Spectral Sequences

3.1 Spectral Sequence associated to a Filtered Differential Module

Let C be a filtered differential module. Our goal is to produce a spectral sequence with E_0 page $\operatorname{Gr} C$ and E_∞ page $\operatorname{Gr}\mathrm{H}C$. For this, for each pair of integers (p, r), let

$$S_r^p := \{x \in F^pC : \partial x \in F^{p+r}C\},$$

which is the collection of elements in F^pC pushed r steps deep into the filtration by ∂.

Remark 3.1. It follows from the definition that

(a) for each $p \in \mathbb{Z}$, we have $S_r^p \supset F^{p+r}C$ for all $r \geq 0$ and $S_r^p = F^pC$ for all $r \leq 0$.
(b) for each $p, r \in \mathbb{Z}$, we have $S_r^p \cap F^{p+1}C = S_{r-1}^{p+1}$, and
(c) for each $p, r, s \in \mathbb{Z}$, we have $\partial S_r^p \subset S_s^{p+r}$.

The idea is that for each $r \geq 0$, the submodule S_r^p behaves like the submodule of r-deep cycles, and its image $\partial S_r^p \subset S_{r+1}^{p+r}$ is behaves the submodule of $(r+1)$-deep coboundaries. To make this precise, consider for each $p \in \mathbb{Z}$ and $r \geq 0$ the submodules of $\operatorname{Gr}^p C$ defined by

$$B_r^p := \frac{\partial S_{r-1}^{p-r+1} + F^{p+1}C}{F^{p+1}C} \subset \frac{S_r^p + F^{p+1}C}{F^{p+1}C} =: Z_r^p,$$

from which we have for each $p \in \mathbb{Z}$ that

$$0 = B_0^p \subset B_1^p \subset B_2^p \subset \cdots \subset Z_2^p \subset Z_1^p \subset Z_0^p = \operatorname{Gr}^p C.$$

For each $p \in \mathbb{Z}$ and $r \geq 0$, let

$$E_r^p := \frac{Z_r^p}{B_r^p} \cong \frac{S_r^p + F^{p+1}C}{\partial S_{r-1}^{p-r+1} + F^{p+1}C} \cong \frac{S_r^p}{\partial S_{r-1}^{p-r+1} + S_{r-1}^{p+1}},$$

where in the last isomorphism we are using Remark 3.1(b), and define the map $\partial_r^p : E_r^p \to E_r^{p+r}$ by the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \partial S_{r-1}^{p-r+1} + S_{r-1}^{p+1} & \longrightarrow & S_r^p & \longrightarrow & E_r^p & \longrightarrow & 0 \\
 & & \downarrow \partial & & \downarrow \partial & & \downarrow \partial_r^p & & \\
0 & \longrightarrow & \partial S_{r-1}^{p+1} + S_{r-1}^{p+r+1} & \longrightarrow & S_r^{p+r} & \longrightarrow & E_r^{p+r} & \longrightarrow & 0,
\end{array}$$

where the rows are short exact; in English, ∂_r^p is given by the recipe "pick representative in S_r^p, take ∂, and then take the class". Let $\pi_r^p : Z_r^p \to E_r^p$ be the canonical surjection. Then it follows immediately for each $r \geq 0$ that $\partial_r^2 = 0$ and for each $p \in \mathbb{Z}$ that

$$B_{r+1}^p = (\pi_r^p)^{-1} B(\partial_r^p) \text{ and } Z_{r+1}^p = (\pi_r^p)^{-1} Z(\partial_r^p),$$

giving us a $\mathbb{Z}$-graded spectral sequence E satisfying $\deg \partial_r = r$ for each $r \geq 0$. This is the required spectral sequence, and this is what allows us to relate $\mathrm{H}^\bullet \operatorname{Gr} C$ and $\operatorname{Gr}^\bullet \mathrm{H}C$. Precisely, we have

Theorem 3.2 (The Fundamental Theorem of Spectral Sequences)**.** Given a filtered differential module C, there is a $\mathbb{Z}$-graded spectral sequence $E = (E_r^\bullet, \partial_r)_{r \geq 0}$, natural in C, such that

(a) for each $r \geq 0$, we have $\deg \partial_r = r$, and
(b) for $r = 0$ we have $E_0 = \operatorname{Gr}^\bullet C$ with the induced differentials (so that $E_1^\bullet = \mathrm{H}^\bullet \operatorname{Gr} C$).

If, further, the filtration on C is both

(i) bounded, i.e., there is a $p_0 \in \mathbb{Z}$ such that $F^pC = 0$ for all $p \geq p_0$, and
(ii) exhaustive, i.e., such that $\bigcup_{p\in\mathbb{Z}} F^pC = C$,

then

(c) the sequence E converges to $\mathrm{Gr}^\bullet \mathrm{H}C$, i.e., $E_\infty^\bullet \cong \mathrm{Gr}^\bullet \mathrm{H}C$.

Further, the convergence is natural in C.

In this case, we say that the sequence E abuts to $\mathrm{H}C$, and write $E \Rightarrow \mathrm{H}C$.

Proof. All that remains to be shown is (c). The conditions (i) and (ii) imply, respectively, that for each $p \in \mathbb{Z}$ we have

$$Z_\infty^p = \frac{F^pZ + F^{p+1}C}{F^{p+1}C} \text{ and } B_\infty^p = \frac{F^pB + F^{p+1}C}{F^{p+1}C}, \tag{1}$$

and so

$$E_\infty^p = \frac{Z_\infty^p}{B_\infty^p} \cong \frac{F^pZ + F^{p+1}C}{F^pB + F^{p+1}C} \cong \frac{F^pZ}{F^pB + F^{p+1}Z} \cong \mathrm{Gr}^p \mathrm{H}C.$$

This isomorphism is given by the recipe "lift to F^pZ and take the image," and is therefore natural in C. ■

That's it! That's all there is to spectral sequences; everything else is addendum. As observed before, Theorem 3.2 is nothing but a (massive) generalization of Theorem 2.3.

3.2 Spectral Sequence associated to a Filtered Cochain Complex

We can combine definitions 2.10 and 2.15 as well, at least for $G = \mathbb{Z}$: if C is a filtered, $\mathbb{Z}$-graded module, then for each $p, q \in \mathbb{Z}$, we define $F^{p,q}C := F^pC^{p+q}$, so that for a fixed $n \in \mathbb{Z}$, the sequence of submodules

$$C^n \supset \cdots \supset F^{p,n-p}C \supset F^{p+1,n-p-1}C \supset \cdots$$

gives a filtration on C^n. In this case, we define the bigraded[5] module associated to C, still denoted by $\operatorname{Gr} C$, by setting

$$\operatorname{Gr}^{p,q} C := \operatorname{Gr}^p C^{p+q} = F^{p,q}C/F^{p+1,q-1}C$$

for $p, q \in \mathbb{Z}$. Similarly, we can combine all three of definitions 2.1, 2.10, 2.15, to arrive at the definition of a filtered cochain complex C: this is a filtered, $\mathbb{Z}$-graded differential module with differential of degree $\deg \partial = 1$ preserving the filtration. In this case, the associated graded module to $\mathrm{H}C$ is also a filtered, $\mathbb{Z}$-graded module with

$$\operatorname{Gr}^{p,q} \mathrm{H}C = \frac{F^p\mathrm{H}^{p+q}C}{F^{p+1}\mathrm{H}^{p+q}C} = \frac{\operatorname{im}(\mathrm{H}^{p+q}F^pC \to \mathrm{H}^{p+q}C)}{\operatorname{im}(\mathrm{H}^{p+q}F^{p+1}C \to \mathrm{H}^{p+q}C)}.$$

In this case, if for each $p, q \in \mathbb{Z}$ and $r \geq 0$, we set

$$S_r^{p,q} = \{x \in F^{p,q}C : \partial x \in F^{p+r,q-r+1}C\},$$

and let

$$B_r^{p,q} := \frac{\partial S_{r-1}^{p-r+1,q+r-2} + F^{p+1,q-1}C}{F^{p+1,q-1}C} \subset \frac{S_r^{p,q} + F^{p+1,q-1}C}{F^{p+1,q-1}C} =: Z_r^{p,q}, \text{ and } E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}},$$

then we obtain a bigrading on the spectral sequence E from the previous section. (The reader is encouraged to convince themselves that the bigrading here, which seems confusing at first glance, is the only natural one possible.) The map $\partial_r^{p,q}$ has to shift p degree by r as before, but only increase the "total" $p+q$ degree by 1, so that $\partial_r^{p,q} : E_r^{p,q} \to E^{p+r,q-r+1}$.The analogous result to Theorem 3.2 is then

Theorem 3.3 (The Fundamental Theorem of Spectral Sequences, Bigraded Version)**.** Given a filtered cochain complex C, there is a bigraded spectral sequence $E = (E_r, \partial_r)_{r\geq 0}$, natural in C, such that

(a) for each $r \geq 0$, we have $\deg \partial_r = (r, -r+1)$, and
(b) for $r = 0$, we have $E_0^{\bullet,\bullet} = \operatorname{Gr}^{\bullet,\bullet} C$ with induced ("vertical") differentials.

If, further, the filtration on C is both

(i) locally bounded, i.e., for each $n \in \mathbb{Z}$ there is an $r_0 \geq 0$ such that $F^rC^n = 0$ for all $r \geq r_0$, and
(ii) locally exhaustive, i.e., such that for each $n \in \mathbb{Z}$ we have $\bigcup_{r\in\mathbb{Z}} F^rC^n = C^n$,

then

(c) the sequence E converges to $\operatorname{Gr}^{\bullet,\bullet} HC$, i.e., $E_\infty^{\bullet,\bullet} \cong \operatorname{Gr}^{\bullet,\bullet} HC$.

Finally, this convergence is natural in C.

[5]Here, and in what follows, "bigraded" simply means graded by $G = \mathbb{Z} \oplus \mathbb{Z}$.

Proof. The proof is identical to that of Theorem 3.2, since the equalities similar to (1) are obtained for a fixed pair (p, q) by applying the new conditions (i) and (ii) to $n = p + q + 1$ and $n = p + q - 1$ respectively–this is why it suffices to work with *locally* bounded and exhaustive filtrations. ■

3.3 Spectral Sequence(s) associated to a Double Complex

One important context in which filtered cochain complexes arise is through double complexes. Let $K^{\bullet,\bullet}$ be a double complex with differentials ∂_v and ∂_h of bidegrees $(1,0)$ and $(0,1)$ satisfying

$$\partial_v^2 = \partial_h^2 = (\partial_v + \partial_h)^2 = 0.$$

Then the total cochain complex $(\mathrm{Tot}(K)^\bullet, \partial)$ defined by

$$\mathrm{Tot}(K)^n = \bigoplus_{p+q=n} K^{p,q}$$

with differential $\partial = \partial_v + \partial_h$ of degree $\deg \partial = 1$ admits a filtration by taking

$$F^r\,\mathrm{Tot}(K)^n = \bigoplus_{\substack{p+q=n\\ p\geq r}} K^{p,q}$$

with $\mathrm{Gr}^{\bullet,\bullet}\,\mathrm{Tot}(K) \cong K^{\bullet,\bullet}$, called for obvious reasons the filtration by columns. This filtered cochain complex gives rise to a bigraded spectral sequence $(E_r^{\bullet,\bullet})_{r\geq 0}$ thanks to Theorem 3.3, with

$$B_r^{p,q} \subset Z_r^{p,q} \subset K^{p,q}$$

and differentials $\partial_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$. The zeroth page E_0 is simply K itself with the vertical differential ∂_v; the first page is the vertical cohomology $\mathrm{H}_v K$ with the horizontal differential induced by ∂_h. In general, we have have for $r \geq 1$ that

$$Z_r^{p,q} = \{x \in K^{p,q} : \partial_v x = 0 \text{ and for } 1 \leq j < r \text{ there are } x_j \in K^{p+j,q-j} \text{ such that }\\ \partial_h x_{j-1} = \partial_v x_j, \text{ where } x_0 = x\}$$

and

$$B_r^{p,q} = \{x \in K^{p,q} : \text{for } 1 \leq j < r \text{ there are } y_j \in K^{p-r+j,q+r-1-j} \text{ such that}\\ \partial_h y_{j-1} = \partial_v y_j, \text{ where } y_0 = 0, \text{ and } \partial_h y_{r-1} = x + \partial_v z \text{ for some } z \in K^{p,q-1}\},$$

with the map $\partial_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ given by the recipe: lift $[x] \in E_r^{p,q}$ to $x \in Z_r^{p,q}$, pick a sequence $(x_j)_{j=1}^{r-1}$ as described, and then $\partial_r^{p,q}[x] = [(-1)^{r-1}\partial_h x_{r-1}]$ in $E_r^{p+r,q-r+1}$.

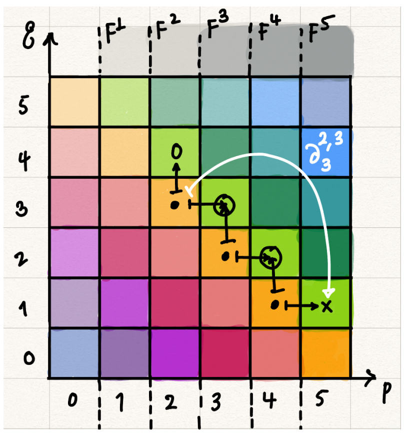


Figure 1: A pictorial representation of the differential $\partial_3^{2,3} : E_3^{2,3} \to E_3^{5,1}$ for a first quadrant double complex. Each color denotes a fixed total degree, and the saturation indicates how deep we are in the filtration.

One sufficient condition that implies the convergence requirements of Theorem 3.3 is that $K^{\bullet,\bullet}$ be a first quadrant double complex, i.e., $K^{p,q} = 0$ if $p < 0$ or $q < 0$. We have proven

Theorem 3.4 (Spectral Sequence of a Double Complex I). Given a double complex $(K^{\bullet,\bullet}, \partial_v, \partial_h)$, there is a bigraded spectral sequence $E = (E_r, \partial_r)_{r\geq 0}$, natural in K, such that

(a) for each $r \geq 0$, we have $\deg \partial_r = (r, -r+1)$,
(b) for $r = 0, 1, 2$, the pages E_r can be described as

$$\begin{aligned} E_0^{p,q} &= K^{p,q} && \text{and } \partial_0^{p,q} = \partial_v, \\ E_1^{p,q} &= \mathrm{H}_v^q(K^{p,\bullet}) && \text{and } \partial_1^{p,q} = [\partial_h], \text{ and} \\ E_2^{p,q} &= \mathrm{H}_h^p \mathrm{H}_v^q(K). \end{aligned}$$

If, further, K is a first quadrant double complex, then

(c) E is a first quadrant spectral sequence that abuts to $\mathrm{H}^\bullet \operatorname{Tot}(K)$.

The theorem is often expressed by writing

$$\mathrm{H}_h^p \mathrm{H}_v^q(K) \Rightarrow \mathrm{H}^{p+q} \operatorname{Tot}(K).$$

The abutment statement means that, once the spectral sequence mechanism has "run", then for each $n \geq 0$, we can recover $\mathrm{H}^n \operatorname{Tot}(K)$ by working "right to left", namely starting with $E_\infty^{n,0}$, which is a subobject of $\mathrm{H}^n \operatorname{Tot}(K)$, and successively solving extension problems moving left and upwards along the diagonal. In other words,

$$F^n \mathrm{H}^n \operatorname{Tot}(K) = E_\infty^{n,0},$$

and for each integer j with $1 \leq j \leq n$, there is a short exact sequence

$$0 \to F^{n-j+1} \mathrm{H}^n \operatorname{Tot}(K) \to F^{n-j} \mathrm{H}^n \operatorname{Tot}(K) \to E_\infty^{n-j,j} \to 0,$$

with $F^0 \mathrm{H}^n \operatorname{Tot}(K) = \mathrm{H}^n \operatorname{Tot}(K)$ at the last stage.

Remark 3.5. If K is a first quadrant double complex, then for each fixed (p,q), the module $E_\infty^{p,q}$ is obtained already at a finite stage; namely, if $r > p$ then $E_r^{p,q} \supset E_{r+1}^{p,q}$ and if $r > q+1$, then $E_r^{p,q} \twoheadrightarrow E_{r+1}^{p,q}$, so that in any case $E_{\max\{p+1,q+2\}}^{p,q} \cong E_\infty^{p,q}$.

Further, we get by taking $q = 0$ that for each n we have

$$E_2^{n,0} \twoheadrightarrow E_3^{n,0} \twoheadrightarrow \cdots \twoheadrightarrow E_{n+1}^{n,0} \xrightarrow{\sim} E_\infty^{n,0},$$

so that the "x-axis" starting on the E_2 pages consists of progressive smaller quotient objects leading to the first graded piece. The resulting map $e_B^n : E_2^{n,0} \twoheadrightarrow E_\infty^{n,0} \hookrightarrow \mathrm{H}^n \operatorname{Tot}(K)$, called the base edge map,[6] can be described as follows: for any $n \geq 0$ and $r \geq 2$, we have

$$Z_r^{n,0} = Z^n(F^n \operatorname{Tot}(K)) = \{x \in K^{n,0} : \partial_v x = \partial_h x = 0\} \subset Z^n \operatorname{Tot}(K).$$

The edge map e_B^n is therefore induced by

$$\begin{array}{ccccc} Z_2^{n,0} & \xrightarrow{\sim} & Z^n F^n \operatorname{Tot}(K) & \hookrightarrow & Z^n \operatorname{Tot}(K) \\ \downarrow & & & & \downarrow \\ E_2^{n,0} & & \dashrightarrow & & \mathrm{H}^n \operatorname{Tot}(K), \end{array}$$

[6] For an explanation of the terminology, see §4.6.1.

or in English by the recipe "lift to $Z_2^{n,0}$ and take image in $\mathrm{H}^n \operatorname{Tot}(K)$".

Similarly, we get by taking $p = 0$ that for each n we have

$$E_1^{0,n} \supset E_2^{0,n} \supset \cdots \supset E_{n+2}^{0,n} = E_\infty^{0,n},$$

so that "y-axis" starting on the E_1 page consists of progressively smaller subobjects leading to the final graded piece. The resulting map $e_F^n : \mathrm{H}^n \operatorname{Tot}(K) \twoheadrightarrow E_\infty^{0,n} \hookrightarrow E_2^{0,n}$, called the fiber edge map, can described by the diagram

$$\begin{array}{ccccc} Z^n \operatorname{Tot}(K) & \twoheadrightarrow & \dfrac{Z^n \operatorname{Tot}(K) + F^1 \operatorname{Tot}(K)^n}{F^1 \operatorname{Tot}(K)^n} = Z_\infty^{0,n} & \hookrightarrow & Z_2^{0,n} \\ \downarrow & & & & \downarrow \\ \mathrm{H}^n \operatorname{Tot}(K) & & \dashrightarrow & & E_2^{0,n}, \end{array}$$

or again in English by "lift to $Z^n \operatorname{Tot}(K)$, take the piece in $K^{0,n}$; this is in $Z_2^{0,n}$, and now take the image in $E_2^{0,n}$".

This "finite-stage stabilization" property of spectral sequences arising from first quadrant double complexes allows us to write down this extension problem of recovering $\mathrm{H}^n \operatorname{Tot}(K)$ for low n explicitly as the exact sequence of low degree terms. Namely,

Proposition 3.6 (Low Degree Exact Sequence)**.** Given a first quadrant double complex K and the clockwise spectral sequence E arising from it, there is an exact sequence

$$0 \to E_2^{1,0} \xrightarrow{e_B^1} \mathrm{H}^1 \operatorname{Tot}(K) \xrightarrow{e_F^1} E_2^{0,1} \xrightarrow{\partial_2^{0,1}} E_2^{2,0} \xrightarrow{e_B^2} \ker(\mathrm{H}^2 \operatorname{Tot}(K) \xrightarrow{e_F^2} E_2^{0,2}) \to E_2^{1,1} \xrightarrow{\partial_2^{1,1}} E_2^{3,0}. \tag{2}$$

For $r \geq 2$, the map $\partial_r^{0,r-1} : E_r^{0,r-1} \to E_r^{r,0}$, is called the transgression map, and is a map from a subobject of $E_2^{0,r-1}$ to a quotient object of $E_2^{r,0}$.

Proof. Follows from stitching together the exact sequences

$$0 \to E_2^{1,0} \xrightarrow{e_B^1} \mathrm{H}^1 \operatorname{Tot} K \to E_\infty^{0,1} \to 0,$$
$$0 \to E_\infty^{0,1} \to E_2^{0,1} \xrightarrow{\partial_2^{0,1}} E_2^{2,0} \to E_\infty^{2,0} \to 0,$$
$$0 \to E_\infty^{2,0} \to F^2\mathrm{H}^2 \operatorname{Tot} K \to E_3^{1,1} \to 0,$$
$$0 \to F^2\mathrm{H}^2 \operatorname{Tot} K \to \mathrm{H}^2 \operatorname{Tot} K \to E_\infty^{0,2} \to 0,$$
$$0 \to E_3^{1,1} \to E_2^{1,1} \xrightarrow{\partial_2^{1,1}} E_2^{3,0},$$

along with noting that in expression of e_F^2 as the composite $\mathrm{H}^2 \operatorname{Tot} K \to E_\infty^{0,2} \hookrightarrow E_2^{0,2}$, the latter map is injective, so that the kernel of e_F^2 is the same as that of $\mathrm{H}^2 \operatorname{Tot} K \to E_\infty^{0,2}$. I will leave it to the reader to fill in the details and verify that the maps are as claimed. ∎

Clearly, reversing the roles of p, q produces another filtration, namely one by rows, and again gives rise to a bigraded sequence, with differentials ∂_r of bidegree $(-r+1, r)$. From this we get

Theorem 3.7 (Spectral Sequence of a Double Complex II). Given a double complex $(K^{\bullet,\bullet}, \partial_v, \partial_h)$, there is a bigraded spectral sequence $E = (E_r, \partial_r)_{r \geq 0}$, natural in K, such that

(a) for each $r \geq 0$, we have $\deg \partial_r = (-r+1, r)$,

(b) for $r = 0, 1, 2$, the pages E_r can be described as

$$\begin{aligned} E_0^{p,q} &= K^{p,q} && \text{and } \partial_0^{p,q} = \partial_h, \\ E_1^{p,q} &= \mathrm{H}_h^p(K^{\bullet,q}) && \text{and } \partial_1^{p,q} = [\partial_v], \text{ and} \\ E_2^{p,q} &= \mathrm{H}_v^q \mathrm{H}_h^p(K). \end{aligned}$$

If, further, K is a first quadrant double complex, then

(c) E is a first quadrant spectral sequence that abuts to $\mathrm{H}^\bullet \operatorname{Tot}(K)$.

This theorem is often expressed by writing

$$\mathrm{H}_v^q \mathrm{H}_h^p(K) \Rightarrow \mathrm{H}^{p+q} \operatorname{Tot} K.$$

The abutment statement means now that to recover $\mathrm{H}^n \operatorname{Tot} K$, we would work "left to right". The rest of the above theory–all of Remark 3.5 and Proposition 3.6–works similarly after swapping the indices p and q everywhere.

When we need to distinguish these, we'll call the sequence arising from the filtration by columns the clockwise spectral sequence and denote it by $\left({}^{(c)}E_r^{\bullet,\bullet}, {}^{(c)}\partial_r^{\bullet,\bullet}\right)_{r\geq 0}$ and the sequence arising from the filtration by rows the counterclockwise spectral sequence, and denote it by $\left({}^{(r)}E_r^{\bullet,\bullet}, {}^{(r)}\partial_r^{\bullet,\bullet}\right)_{r\geq 0}$. Both of these sequences abut to the same object–namely the total cohomology $\mathrm{H}^\bullet \operatorname{Tot} K$–and the power of spectral sequences often lies in making the resulting comparisons.

3.4 Product Structures in Spectral Sequences*

From filtered DGA.

4 Examples

4.1 Some Named Algebraic Lemmas

In this section, we will use the above theory to easily deduce some well-known named lemmas, other proofs of which using diagram chasing are quite tedious and messy.

4.1.1 Snake Lemma

The first standard consequence is the Snake Lemma.

Lemma 4.1 (Snake). Given a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & & \\ & & \uparrow{\scriptstyle\alpha} & & \uparrow{\scriptstyle\beta} & & \uparrow{\scriptstyle\gamma} & & \\ & & A & \xrightarrow{f} & B & \xrightarrow{g} & C & \longrightarrow & 0 \end{array}$$

with exact rows, there is an exact sequence

$$0 \to \ker f \to \ker \alpha \xrightarrow{[f]} \ker \beta \xrightarrow{[g]} \ker \gamma \xrightarrow{\delta} \operatorname{coker} \alpha \xrightarrow{[f']} \operatorname{coker} \beta \xrightarrow{[g']} \operatorname{coker} \gamma \to \operatorname{coker} g' \to 0.$$

The map $\ker f \to \ker \alpha$ is the inclusion map, and the map $\operatorname{coker} \gamma \to \operatorname{coker} g'$ comes from the inclusion $\gamma(C) \subset g'(B')$. The map $[f]$ is the map induced by f, and similarly for g, f', and g'.

Proof. Consider the above diagram as a first quadrant double complex with A in position $(0, 0)$. On the one hand, the associated clockwise spectral sequence has E_1 page

$$\begin{array}{ccccc} \operatorname{coker} \alpha & \xrightarrow{[f']} & \operatorname{coker} \beta & \xrightarrow{[g']} & \operatorname{coker} \gamma \\ \\ \ker \alpha & \xrightarrow{[f]} & \ker \beta & \xrightarrow{[g]} & \ker \gamma \end{array}$$

and E_2 page

$$\begin{array}{ccc} \ker[f'] & * & \operatorname{coker}[g'] \\ & \searrow^{\partial} & \\ \ker[f] & * & \operatorname{coker}[g] \end{array}$$

for some map ∂ and some objects in the places marked with a $*$. The only difference in the $E_3 = E_\infty$ page is that $\ker[f']$ (resp. $\operatorname{coker}[g]$) gets replaced by $\ker \partial$ (resp. $\operatorname{coker} \partial$). On the other hand, the associated counterclockwise spectral sequence has all zeroes on the $E_1 = E_\infty$ page except for $\ker f$ at $(0, 0)$ and $\operatorname{coker} g'$ at $(2, 1)$. Comparing degree n terms in total cohomology for $n = 0, 1, 2, 3$ yields, respectively that

0. $\ker[f] = \ker f$,[7]

[7] Here and below, an equality sign means that the natural map between the two objects on either side is an isomorphism.

1. the lower entry labelled $*$ is zero and ∂ is a monomorphism,
2. the upper entry labelled $*$ is zero and ∂ is an epimorphism, and
3. $\operatorname{coker}[g'] = \operatorname{coker} g'$.

In all, ∂ is an isomorphism; let δ denote the composite $\ker\gamma \twoheadrightarrow \operatorname{coker}[g] \xrightarrow{\partial^{-1}} \ker[f'] \hookrightarrow \operatorname{coker}\alpha$. The result then follows from putting all the above results together. ∎

We leave to the reader the task of checking that "snake map" δ constructed above agrees with the one they may have seen constructed in a different way (say using diagram-chasing).

Corollary 4.2 (Kernel-Cokernel Exact Sequence)**.** Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a composable pair of morphisms. Then there is an exact sequence

$$0 \to \ker f \to \ker gf \xrightarrow{[f]} \ker g \to \operatorname{coker} f \xrightarrow{[g]} \operatorname{coker} gf \to \operatorname{coker} g \to 0.$$

Again, the square brackets denote induced maps. The map $\ker g \to \operatorname{coker} f$ is the composite $\ker g \hookrightarrow B \twoheadrightarrow \operatorname{coker} f$.

Proof. Apply Lemma 4.1 to

$$\begin{array}{ccccccc}
0 & \longrightarrow & C & \xrightarrow{\mathrm{id}_C} & C & \longrightarrow & 0 & & \\
 & & {\scriptstyle gf}\uparrow & & {\scriptstyle g}\uparrow & & \uparrow & & \\
 & & A & \xrightarrow{f} & B & \longrightarrow & \operatorname{coker} f & \longrightarrow & 0.
\end{array}$$

It remains to identify the "snake map" δ of Lemma 4.1 with the map $[g] : \operatorname{coker} f \to \operatorname{coker} gf$ induced by g, a task we leave to the reader. ∎

Remark 4.3. The Kernel-Cokernel Exact Sequence is responsible for the additivity of the Herbrand quotient as well as of the index of Fredholm operators.

4.1.2 Weak Four and Five Lemmas

Lemma 4.4 (Weak Four)**.** Suppose are given a commutative diagram

$$\begin{array}{ccccccc}
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D \\
{\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & {\scriptstyle\delta}\uparrow \\
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D
\end{array}$$

with exact rows.

(a) If α is epic, and β and δ monic, then γ is monic.
(b) If δ is monic, and α and γ are epic, then β is epic.

This can be deduced directly from diagram chasing or from repeated applications of the Snake Lemma (Lemma 4.1)–both proofs left to the reader–but there is a better way to argue directly using spectral sequences as follows.

Proof. We'll prove (a); then (b) follows either by duality or by a similar argument. Consider the above diagram as a first quadrant double complex with A at position $(0,0)$. On the one hand, the clockwise spectral sequence has E_1 page

$$\begin{array}{cccc} 0 & \operatorname{coker}\beta \xrightarrow{[g']} & \operatorname{coker}\gamma \xrightarrow{[h']} & \operatorname{coker}\delta \\ \\ \ker\alpha & 0 & \ker\gamma & 0, \end{array}$$

where again square brackets denote induced maps. There are no further differentials coming in or out of the entry $\ker\gamma$ in position $(2,0)$, and hence $E_\infty^{2,0} = \ker\gamma$. On the other hand, the counterclockwise spectral sequence evidently has zeroes at the positions that could potentially contribute to total cohomology in degree 2, and so this forces $\ker\gamma = 0$. ■

Corollary 4.5 (Five Lemma)**.** Suppose we are given a commutative diagram

$$\begin{array}{ccccccccc} A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D' & \xrightarrow{i'} & E' \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow & & {\scriptstyle\gamma}\uparrow & & {\scriptstyle\delta}\uparrow & & {\scriptstyle\varepsilon}\uparrow \\ A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D & \xrightarrow{i} & E \end{array}$$

with exact rows. If β and δ are isomorphisms, α is epic, and ε is mono, then γ is an isomorphism.

Proof. Follows from two applications of Lemma 4.4, or directly via a very similar (and in fact easier) spectral sequence argument. ■

4.1.3 The n^2-Lemma for $n \geq 2$

Similarly, we can now prove

Lemma 4.6 (The n^2-Lemma)**.** Let $n \in \mathbb{Z}_{\geq 2}$, and suppose we are given a commutative $n \times n$ grid with rows and columns complexes, and all rows, and all columns but possibly one, exact. Then all rows and columns are exact.

The same result holds (say by symmetry) all columns, and all rows but possibly one, are exact.

Proof. Immediate from comparing the abutment of the clockwise and counterclockwise spectral sequences. ■

In fact, the spectral sequence machinery also gives us the tools to very easily prove Maclane's variations of the nine lemma.

Lemma 4.7. Sharp

Lemma 4.8. Symmetric

It is left up to the reader to come up with more such lemmas and to prove them using the machinery of spectral sequences.

4.2 Spectral Sequences Arising from a Filration on the Space

4.2.1 Equivalence of Cellular and Singular Homology

4.2.2 Serre Spectral Sequence via CW Complexes

As an example, we compute the cohomology of $\mathrm{K}(\mathbb{Z}, 3)$.

4.3 Künneth and Universal Coefficient Spectral Sequences

4.4 Čech-de Rham Spectral Sequence

4.4.1 Mayer-Vietoris Sequence

4.4.2 An Application to Symplectic Topology

4.5 Frölicher Spectral Sequences

4.5.1 Compact Kähler Manifolds

4.5.2 Stein Manifolds

4.6 Grothendieck Spectral Sequence

4.6.1 Leray-Serre Spectral Sequence

4.6.2 Čech-Derived Functor Cohomology Spectral Sequence

4.6.3 Lyndon-Hochschild-Serre Spectral Sequence

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