# Indra's Pearls

## Symmetry, Recursion, and Beauty in Mathematics

Dhruv Goel

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# Symmetry



What is symmetry?

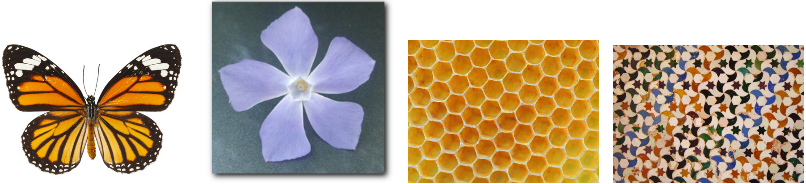


Figure: Some symmetric configurations. Src: Google Images.

A symmetry of a configuration is a transformation that preserves a special property.

**Definition**

A *symmetry* is a triple $(X, \mathscr{P}, T)$, where $X$ is a set, $\mathscr{P}$ is a "property" of $X$, and $T : X \to X$ a bijective transformation preserving $\mathscr{P}$.

(1) E.g. $\mathscr{P} : X \to C$ and $T$ preserves $\mathscr{P}$ iff $\mathscr{P} \circ T = \mathscr{P}$.

(2) Given $(X, \mathscr{P})$, the set of all $T$ preserving $\mathscr{P}$ forms a group under composition: if $T, S$ preserve $\mathscr{P}$, then so do $T \circ S$ and $T^{-1}$. This group is denoted by $\mathrm{Aut}_{\mathscr{P}}(X)$, and is called the *symmetry group* of the configuration $(X, \mathscr{P})$.[1]

[1] This modern perspective has origins in the *Erlangen Program* of Felix Klein (1849-1925).

# Examples

## Example

Take $X = \mathbb{R}$ and $\mathscr{P} : X \to \mathbb{Z}$ by $x \mapsto \lfloor x \rfloor$, and take $T_n(x) = x + n$ for $n \in \mathbb{Z}$. In this case, $\mathrm{Aut}_{\mathscr{P}}(X) = \mathbb{Z}$.

## Example

If $X$ is the flower on the previous slide, and $\mathscr{P}$ its shape, then $\mathrm{Aut}_{\mathscr{P}}(X) = \mathbb{Z}/5$.

## Example

If $X$ is an equilateral triangle and $\mathscr{P}$ its rigid shape, then

$$\mathrm{Aut}_{\mathscr{P}}(X) = S_3 = \langle \sigma, \tau | \sigma^2, \tau^3, \sigma\tau\sigma^{-1}\tau \rangle.$$

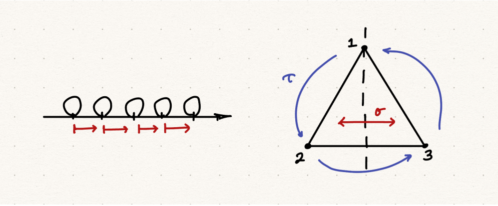



Figure: Symmetry groups $\mathbb{Z}$ and $S_3$. Picture made with Notability.

# Euclidean Symmetries

**Definition**

A *Euclidean symmetry* (or *rigid symmetry*) of $X = \mathbb{R}^2$ is a $T : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves distances.

In other words, $\mathscr{P} : X \times X \to \mathbb{R}_{\geq 0}$ by $(x, y) \mapsto |x - y|$, and we ask that for all $x, y \in X$ we have

$$\mathscr{P}(Tx, Ty) = \mathscr{P}(x, y).$$

A Euclidean symmetry also preserves (unoriented) angles by

$$\cos\theta = \frac{\langle v, w \rangle}{|v| \cdot |w|}.$$

In this case, $\mathrm{Aut}_{\mathscr{P}}(X) = \mathrm{Euc}(\mathbb{R}^2) \cong \mathbb{R}^2 \rtimes \mathrm{O}_2$. This includes

(1) translations,
(2) rotations,
(3) reflections, and
(4) their compositions.

These are all!

# Conformal Symmetries

**Definition**

A *conformal symmetry* of $\mathbb{R}^2$ is a transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that preserves angles.

Think of $\mathbb{R}^2 = \mathbb{C}$; then all transformations are of the form

$$T_{a,b} : z \mapsto az + b$$

for $a, b \in \mathbb{C}$ with $a \neq 0$.

One other operation: $z \mapsto 1/z$, which is inversion in unit circle followed by reflection.

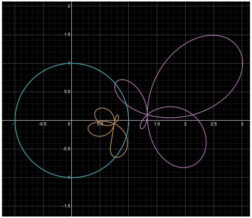


Figure: The map $z \mapsto 1/z$ on $\mathbb{C}$. Picture made with Desmos.

Except the one point $z = 0$: something weird happens there!

# Riemann Sphere and Möbius Transformations



**Definition**

The *Riemann sphere* is the space $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} = \mathbb{P}^1_{\mathbb{C}}$.

This is a sphere via *stereographic projection*, which is conformal and takes circles to circles.

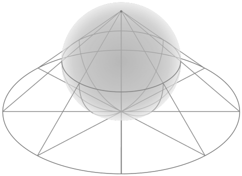


Figure: The Riemann Sphere. Src: Google Images.

**Definition**

A *Möbius transformation* is a conformal symmetry of $\hat{\mathbb{C}}$.[2]

The symmetry group is $\mathrm{PSL}_2\,\mathbb{C} = \mathrm{PGL}_2\,\mathbb{C}$, which consists of transformations of the form

$$z \mapsto \frac{az+b}{cz+d}.$$

[2]Named after German mathematician August Ferdinand Möbius (1790-1868).

# Types of Möbius Transformations

There are four types of Möbius transformations:

(a) elliptic, e.g. $z \mapsto e^{2\pi i/3}z$,
(b) hyperbolic, e.g. $z \mapsto 2z$,
(c) loxodromic, e.g. $z \mapsto (1+i)z$, and
(d) parabolic, e.g. $z \mapsto z+1$.

A parabolic transformation has one fixed point; the others have two each.

For loxodromic transformations, there is a **source** and a **sink**.

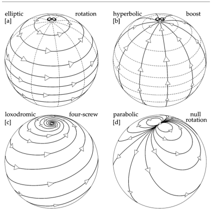
Examples:

(1) The transformation $z \mapsto 1/z$ is elliptic with fixed points $\pm 1$.
(2) The transformation $T$ given by

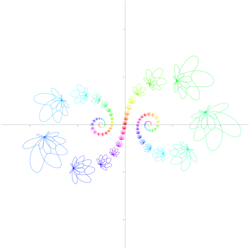
$$T : z \mapsto \frac{(1-5i)z+1}{z+(1-5i)}$$

is loxodromic with src. $-1$ and sink $1$.

For more on this, see Needham's *Visual Complex Analysis*, [Nee97].



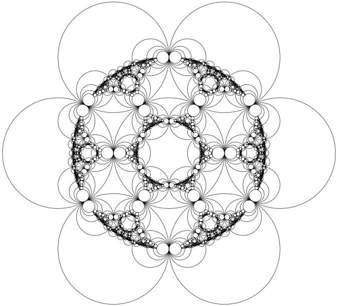

(a) Src: [Nee97, Img. 3.26].

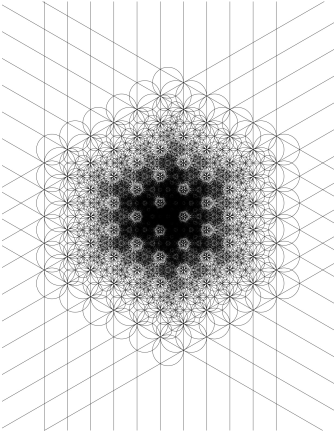


(b) The lox. trans. $T$. Src. Sage.

# Iteration and Recursion

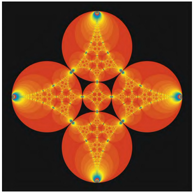
What happens if you iterate pairs of transformations? Groups of them?
The following pictures were made with `lim`, a program by Curt McMullen [McM].



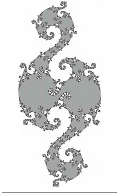


# Examples: Limits of Kleinian Groups

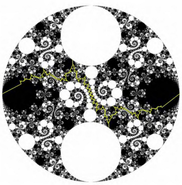




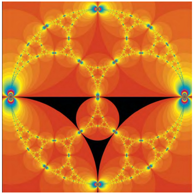
(a) [MSW02, p. xviii].



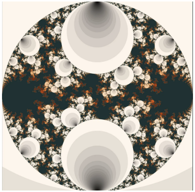
(b) [MSW02, Fig. 8.1].



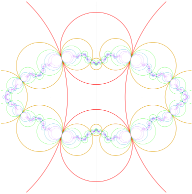
(c) [MSW02, Fig. 10.6].



(d) [MSW02, p. 200].



(e) Src: [Belb].



(f) Fuchsian. Src: Sage.

Figure: Some more limits of Kleinian groups.

See *Indra's Pearls: The Vision of Felix Klein* by Mumford-Series-Wright [MSW02] the fantastic visualization [Bela] and its explanation [Belb].

# The Pursuit of Beauty

What is beauty?

Beauty can be...
(1) inexplicable,
(2) elusive,
(3) unexpected, and
(4) exciting.
Claim: only essential feature is (4).

Art is an expression of beauty.

Math is a form of art.

In *Mathematics for Human Flourishing* [SJ20], Su classifies mathematical beauty into four types: it can be
(1) sensory or visual,
(2) wondrous,
(3) insightful, or
(4) transcendent.

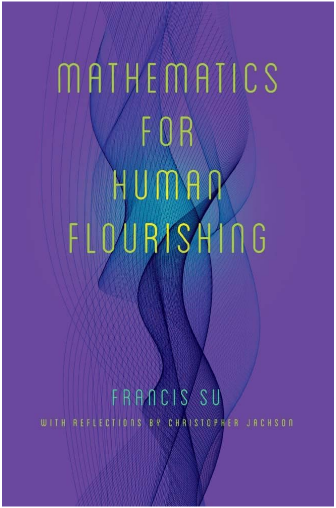



Figure: *Mathematics for Human Flourishing* by Su and Jackson, [SJ20].

# Takeaways: Visual Beauty, Infinity, and Fractals

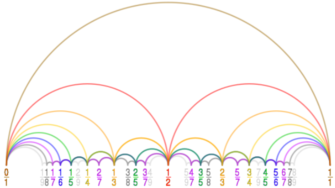


Visual beauty is intimately related to two things: **symmetry** and **chaos**.

These often combine in very fruitful ways, often via *recursion*–this leads to $\infty$ and **fractals**.

We live in an age where we all have access to very powerful technology that can help us understand and appreciate this beauty like never before.

All of math is very intricately interconnected–go and learn it all!



(a) Src: Wikipedia.

(b) Src: Wikipedia

# References



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