Hamiltonian Actions and Marsden-Weinstein-Meyer Reduction

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Abstract

This paper is written in fulfillment of the requirements of a Math 91R (supervised reading and research) course offered at Harvard during the spring of 2022 on symplectic geometry by Prof. Denis Auroux. In this paper, I give explicit computations of moment maps for several examples, explain the thoery of symplectic reduction, and compute a couple of concrete examples of reduced spaces. The primary reference for these examples is Homework 19 and 20 from da Silva's *Lectures on Symplectic Geometry* [1].

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1 Introduction to Hamiltonian Actions and Noether's Theorem

In this section, we present the basic definitions and the modern mathematical formulation of Emmy Noether's theorem relating symmetries and integrals of motion. We follow [1], Chapter 22 and Section 24.1.

Definition 1. Let (M,ω) be a symplectic manifold, G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^* = \mathrm{Hom}_{\mathbb{R}}(\mathfrak{g},\mathbb{R})$ its dual. A smooth action $\rho : G \to \mathrm{Symp}(M,\omega)$ of G on M by symplectomorphisms is called *Hamiltonian* if there is a smooth map

$$\mu : M \to \mathfrak{g}^*$$

called the *moment map* such that

- for each $X \in \mathfrak{g}$, the smooth map $\mu^X : M \to \mathbb{R}$ given by $\mu^X(p) := \langle \mu(p), X\rangle$ for $p \in M$ is a Hamiltonian function for the vector field $\underline{X}$ on M which is the infinitesimal generator of the 1-parameter subgroup $\{\rho(\exp tX)\}_{t\in\mathbb{R}}$ of symplectomorphisms of M, i.e. that $\underline{X} \lrcorner\, \omega = \mathrm{d}\mu^X$, and
- the map μ is G-equivariant for ρ and the coadjoint action on $\mathfrak{g}^*$.

This setup (M,ω,G,ρ,μ) is called a *Hamiltonian G-space*.

Note that when $G = \mathbb{R}$ or S^1, then this recovers the definition of a Hamiltonian function. There are various ways of formulating the equivariance condition. It says that for any $p \in M$ and $g \in G$, we have

$$\mu(gp) = \mathrm{Ad}_g^* \mu(p),$$

or equivalently that for any $g \in G$ and $Y \in \mathfrak{g}$ we have that

$$\mu^{\mathrm{Ad}_g Y} = \mu^Y \circ g^{-1}.$$

Taking $g = \exp(tX)$ for some $X \in \mathfrak{g}$ and differentiating at $t = 0$, we conclude that for all X, Y

$$\mu^{[X,Y]} = \mu^{\mathrm{ad}_X Y} = -\underline{X}\mu^Y = -\underline{X} \lrcorner\, \underline{Y} \lrcorner\, \omega = \omega(\underline{X},\underline{Y}) = \{\mu^X,\mu^Y\}.$$

The map $\mu^* : \mathfrak{g} \to \mathscr{C}^\infty(M)$ given by $X \mapsto \mu^X$ has therefore the property that

- for each $X \in \mathfrak{g}$, the function μ^X is Hamiltonian for $\underline{X}$, and
- the map μ^* is a Lie algebra homomorphism for the Lie bracket on $\mathfrak{g}$ and the Poisson bracket on $\mathscr{C}^\infty(M)$.

Such a map is called a *comoment map*. We have shown that a moment map gives rise naturally to a comoment map. The converse is also true, at least when G is connected. Since we will not really need this, we refer the reader to [2], Lemma 5.2.1 for a proof.

Here we make two simple observations, whose proofs are immediate, and which will be handy below.

Observation 1. If (M,ω,G,ρ,μ) is a Hamiltonian G-space, and $H \subset G$ an (embedded) Lie subgroup, then $(M,\omega,H,\rho|_H,\mathrm{res}\circ\mu)$ is a Hamiltonian H-space, where $\mathrm{res}:\mathfrak{g}^* \to \mathfrak{h}^*$ is simply the restriction map.

Observation 2. If $(M_i,\omega_i,G,\rho_i,\mu_i)$ are Hamiltonian G-spaces for a fixed G and $i = 1,\dots,n$, then

$$\left(\prod_i M_i, \sum_i \mathrm{pr}_i^*\,\omega_i, G, \gamma\circ\prod_i \rho_i\circ\Delta, \sum_i \mu_i\circ\mathrm{pr}_i\right)$$

is a Hamiltonian G-space, where $\Delta : G \to G^n$ is the diagonal embedding, and γ is the obvious map

$$\prod_i \mathrm{Symp}(M_i,\omega_i) \hookrightarrow \mathrm{Symp}(\prod_i M_i, \sum_i \mathrm{pr}_i^*\,\omega_i).$$

Symplectic geometry has its origins in the study of classical mechanics, in which Emmy Noether's theorem is a fundamental theorem relating symmetries and integrals of motion of a physical system. The modern formulation of Noether's Theorem relates G-invariance of smooth functions $f : M \to \mathbb{R}$ to the values of the moment map μ on the trajectories of the Hamiltonian vector field X_f of f (which, as the reader will recall, is defined uniquely by $X_f \lrcorner\, \omega = \mathrm{d}f$).

Theorem 1 (Noether's Theorem)**.** Let (M,ω,G,ρ,μ) be a Hamiltonian G-space. If a smooth function $f : M \to \mathbb{R}$ is G-invariant, then μ is constant along the trajectories of the Hamiltonian vector field X_f of f. The converse holds if G is connected.[1]

Proof. This follows from the Cartan's magic formula computation

$$\mathscr{L}_{X_f}\mu^X = X_f \lrcorner\, \mathrm{d}\mu^X = X_f \lrcorner\, \underline{X} \lrcorner\, \omega = -\underline{X} \lrcorner\, X_f \lrcorner\, \omega = -\underline{X} \lrcorner\, \mathrm{d}f = -\mathscr{L}_{\underline{X}} f$$

for any $X \in \mathfrak{g}$. Therefore, if f is G-invariant, then $\mathscr{L}_{\underline{X}} f = 0$ for all $X \in \mathfrak{g}$ and hence $\mathscr{L}_{X_f}\mu^X = 0$ for all $X \in \mathfrak{g}$, proving that μ is constant along the trajectories of X_f. Conversely, if μ is constant along the trajectories of X_f, then $\mathscr{L}_{X_f}\mu^X = 0$ for all X, so that $\mathscr{L}_{\underline{X}} f = 0$ for all X. Therefore, the proof follows from the following lemma:

Lemma 1. Let G be a connected Lie group acting smoothly a manifold M. Then a function $f : M \to \mathbb{R}$ is G-invariant iff $\mathscr{L}_{\underline{X}} f = 0$ for all $X \in \mathfrak{g}$.

Proof. One direction is clear; for the other, take $Z := \{g \in G : \forall\, p \in M, f(gp) = f(p)\} \subset G$. Clearly $e \in Z$, the subset Z is closed, and has the property that if $U \subset Z$ is any subset, then for all $g \in Z$ we have $gU \subset Z$. Therefore, it suffices to show that Z contains a neighborhood of e. We know that the exponential map $\exp : \mathfrak{g} \to G$ maps a neighborhood V of 0 diffeomorphically onto a neighborhood U of e in G. We claim that $U \subset Z$. Indeed, take $g \in U$ so that $g = \exp(X)$ for some $X \in V$. Define a function $h : \mathbb{R}\times M \to \mathbb{R}$ by $(t,p) \mapsto f(\exp(tX)p)$. Then for all (t,p) we have $h(t+\varepsilon,p) = h(\varepsilon,\exp(tX)p) + h(t,p)$, so that $\partial_t h|_{(t,p)} = \mathscr{L}_{\underline{X}}(f)(\exp(tX)p) = 0$ by hypothesis. This implies that $\partial_t \lrcorner\, \mathrm{d}h = 0$, so that h is constant along the trajectories of ∂_t. In particular, for any $p \in M$, we have that $f(gp) = f(\exp(1X)p) = f(p)$ as needed. ■

■

Definition 2. Let (M,ω,G,ρ,μ) be a Hamiltonian G-space.

- An *integral of motion* of this Hamiltonian G-space is a smooth G-invariant function $f : M \to \mathbb{R}$.
- A *symmetry* of this Hamiltonian G-space is the one parameter subgroup of diffeomorphisms corresponding to a Hamiltonian vector field X, along whose trajecories the moment map is constant.

The Noether principle asserts that (at least when G is connected), there is a bijection between integrals of motion and symmetries of a Hamiltonian G-space.

2 A Plethora of Examples

These examples are taken from Homework 19 of [1]. A general class of examples is related to the following observation:

[1] The reference [1] in Theorem 24.1 does not mention the connectedness assumption, but it is in fact necessary. For a counterexample when G is not connected, take $G = \mathbb{R}^2 \times \{\pm 1\}$ acting on $M = G$, i.e. itself by left multiplication. Then G acts by symplectomorphisms for $\omega := p_1^*(\mathrm{d}x \wedge \mathrm{d}y)$, and we may consider the function $f : G \xrightarrow{p_2} \{\pm 1\} \hookrightarrow \mathbb{R}$, where p_i denotes projection onto the i^{th} factor for $i = 1,2$. Then it is easy to see that this action is Hamiltonian, and of course any μ is constant along the trajectories of the vector field $X_f = 0$, but f is clearly not G-invariant, since G acts on itself transitively and f is not constant.

Observation 3. If a Lie group G acts smoothly on a manifold M, then G acts smoothly on its tangent bunlde TM by the formula $(g,(q,v)) \mapsto (gq, \mathrm{d}g_q v)$ and on its cotangent bundle[2] T^*M by $(g,(q,p)) \mapsto (gq, {}^t\mathrm{d}g_q^{-1}p)$. If $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ is the canonical symplectic form on T^*M, then its pullback under the G-action that arises as above is given in local coordinates by $g^*\omega = \mathrm{d}(q^j g) \wedge ({}^t\mathrm{d}g_q^{-1})^i_j \mathrm{d}p_i$.

Example 1. Consider the action of $G = \mathbb{R}^n$ on $M = \mathbb{R}^n$ by translations; this gives rise to an action on the cotangent bundle (T^*M, ω) by $(t,(q,p)) \mapsto (q+t, p)$, and this is clearly an action by symplectomorphisms. Here, $\mathfrak{g} = \mathbb{R}\langle \frac{\partial}{\partial t^i} \rangle$, and $\mathfrak{g}^* = \mathbb{R}\langle \mathrm{d}t^i \rangle$. If $X = \lambda^i \frac{\partial}{\partial t^i} \in \mathfrak{g}$, then the vector field $\underline{X} = \lambda^i \frac{\partial}{\partial q^i}$ and $\underline{X} \lrcorner\, \omega = \lambda^i \mathrm{d}p_i$. Therefore, if we take $\mu : M \to \mathfrak{g}^*$ to be given by $\mu(q,p) = p_i \mathrm{d}t^i$, then $\mu^X(q,p) = \lambda^i p_i$ and $\mathrm{d}\mu^X = \lambda^i \mathrm{d}p_i = \underline{X} \lrcorner\, \omega$. Since G is abelian, the Ad-equivariance condition simply becomes G-invariance, which this μ certainly satisfies. It follows that this action is Hamiltonian with moment map μ.

Example 2. Consider the standard action of $G = \mathrm{SO}_n\mathbb{R}$ on $M = \mathbb{R}^n$; this gives rise to an action on the tangent bundle TM as explained above. In local coordinates (q,v), this action is given simply by $(g,(q,v)) \mapsto (gq, gv)$.

Now we use the tangent-cotangent isomorphism given by the Euclidean inner product on $\mathbb{R}^n$ to produce a symplectic form on TM corresponding to the canonical form on T^*M; in local coordinates (q^i, v^i), this is given by[3] $\omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}v^i$. It follows that for $g \in \mathrm{SO}_n\mathbb{R}$, we have

$$g^*\omega = \sum_{i,j,k} g^i_j \mathrm{d}q^i \wedge g^i_k \mathrm{d}v^k = \sum_{j,k}\left(\sum_i g^i_j g^i_k\right) \mathrm{d}q^j \wedge \mathrm{d}v^k.$$

But now since $g \in \mathrm{SO}_n\mathbb{R}$, we have that $\sum_i g^i_j g^i_k = \sum_i (g^t)^j_i g^i_k = (g^t g)^j_k = \delta^j_k$, and we conclude that $g^*\omega = \omega$. Therefore, this is an action by symplectomorphisms.

The Lie algebra $\mathfrak{g} = \mathfrak{so}_n\mathbb{R} = \{X \in \mathfrak{gl}_n\mathbb{R} : X + {}^tX = 0\}$ and the dual $\mathfrak{g}^* = \mathfrak{so}_n^*\mathbb{R} = \mathbb{R}\langle \mathrm{d}x^i_j \rangle / \langle \mathrm{d}x^i_j + \mathrm{d}x^j_i \rangle$. If $X = (X^i_j) \in \mathfrak{so}_n\mathbb{R}$, then the vector field $\underline{X} = \sum_{i,j}\left(X^i_j q^j \frac{\partial}{\partial q^i} + X^i_j v^j \frac{\partial}{\partial v^i}\right)$, so that $\underline{X} \lrcorner\, \omega = \sum_{i,j,k} X^k_j (q^j \mathrm{d}v^k - v^j \mathrm{d}q^k)$. Therefore, if $\mu : T\mathbb{R}^n \to \mathfrak{so}_n^*\mathbb{R}$ is the map given by $\mu(q,v) = \sum_{j<k} \mathrm{d}x^k_j (q^j v^k - v^j q^k)$, then μ satisfies that $\underline{X} \lrcorner\, \omega = \mathrm{d}\mu^X$ for all $X \in \mathfrak{so}_n\mathbb{R}$. The checking of Ad-invariance is reduced to showing that for all $g \in \mathrm{SO}_n\mathbb{R}$, and $X \in \mathfrak{so}_n\mathbb{R}$ and $(q,v) \in T\mathbb{R}^n$ we have that $\mu^{gXg^{-1}}(q,v) = \mu^X(g^{-1}q, g^{-1}v)$, which is straightforward but tedious and hence omitted. It follows that the action of $\mathrm{SO}_n\mathbb{R}$ on $T\mathbb{R}^n$ is Hamiltonian with moment map μ.

In the previous two examples, we see that the actions of $\mathbb{R}^n$ by translations and $\mathrm{SO}_n\mathbb{R}$ by rotations on $\mathbb{R}^n$ gives rise to momentum maps which are basically the classical linear and angular momentums respectively. This is the origin on the term "moment map".

Example 3. Fix[4] an n-tuple of integers $k := (k_1, \ldots, k_n)$, and consider the action ρ_k of $\mathbb{T}^n$ on $\mathbb{C}^n$ by

$$(t_1, \ldots, t_n) \cdot (z_1, \ldots, z_n) = (t_1^{k_1} z_1, \ldots, t_n^{k_n} z_n).$$

If $\mathbb{C}^n$ is given the usual symplectic form $\omega = \frac{i}{2}\sum_i \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_i = \sum_i r_i \mathrm{d}r_i \wedge \mathrm{d}\theta_i$, then any fixed $t \in \mathbb{T}^n$ acts only by shifting the θ_i linearly and hence acts as a symplectomorphism. Note that the Lie algebra $\mathfrak{t}^n = \mathbb{R}\langle \frac{\partial}{\partial t_i} \rangle$ and $(\mathfrak{t}^n)^* = \mathbb{R}\langle \mathrm{d}t^i \rangle$. If $X = \sum_i v_i \frac{\partial}{\partial t_i} \in \mathfrak{t}^n$, then the vector field $\underline{X} = -\sum_i v_i k_i \frac{\partial}{\partial \theta_i}$, and so $\underline{X} \lrcorner\, \omega = -\sum_i v_i k_i r_i \mathrm{d}r_i$. Therefore, if $\mu_k : \mathbb{C}^n \to (\mathfrak{t}^n)^*$ is given by $\mu_k(z_1, \ldots, z_n) = -\frac{1}{2}\sum_i k_i |z_i|^2 \mathrm{d}t_i + C$ for any constant $C \in (\mathfrak{t}^n)^*$, then μ_k satisfies that for any $X \in \mathfrak{t}^n$ we have $\underline{X} \lrcorner\, \omega = \mathrm{d}\mu_k^X$; it is also clearly $\mathbb{T}^n$-invariant (and hence Ad-equivariant since $\mathbb{T}^n$ is abelian), and therefore this action is Hamiltonian with moment map μ_k.

Example 4. Consider the Hamiltonian $\mathbb{T}^n$-space $(\mathbb{C}^n, \omega, \mathbb{T}^n, \rho_k, \mu_k)$ of the previous example. Note that the diagonal map $\Delta : S^1 = \mathbb{T}^1 \to \mathbb{T}^n$ gives an embedded Lie subgroup of $\mathbb{T}^n$; the restriction map $(\mathfrak{t}^n)^* \to \mathfrak{t}^*$ is given simply by sending each $\mathrm{d}t_i \mapsto \mathrm{d}t$. Now using the first observation above, we conclude that we get a Hamiltonian S^1-space $(\mathbb{C}^n, \omega, S^1, \rho_k|_{S^1}, \mathrm{res} \circ \mu_k)$, where unpacking the definition shows that the moment map $\mathrm{res} \circ \mu_k : \mathbb{C}^n \to \mathfrak{t}$ is the map $(z_1, \ldots, z_n) \mapsto \left(-\frac{1}{2}\sum_i k_i |z_i|^2\right) \mathrm{d}t + C$ for any $C \in \mathfrak{t}^*$. In particular, taking $k = (1, \ldots, 1)$, the action of S^1 on $\mathbb{C}^n$ by left multiplication is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{t} \cong \mathbb{R}$ given by $(z_1, \ldots, z_n) \mapsto -\frac{1}{2}\sum_i |z_i|^2 + C$ for any constant $C \in \mathbb{R}$. It is often convenient (as we shall see below in Example 8) to take $C = 1/2$.

Example 5. Consider the standard action of $G = \mathrm{Sp}_{2n}\mathbb{R}$ on $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$ (where $\omega_{\mathrm{std}}(v,w) = v^t J w$ for $J = \begin{bmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{bmatrix}$ the usual almost complex structure), which is an action by symplectomorphisms by definition. We know that $\mathfrak{g} = \mathfrak{sp}_{2n}\mathbb{R} = \{X \in \mathfrak{gl}_n\mathbb{R} : X^t J + JX = 0\}$. Consider the map $\mu : \mathbb{R}^{2n} \to \mathfrak{sp}_{2n}^*\mathbb{R}$ given by

$$\mu(v) : X \mapsto -\frac{1}{2}(Xv)^t J v.$$

[2] Here, we put the letter t denoting transpose on the left to avoid a cluttering or too many parentheses.

[3] In this example only, Einstein summation convention gets a little wonky because of the Euclidean inner product, so we'll write down the summation symbols explicitly.

[4] In this example and the next one, for the sake of clarity, we ditch the Einstein convention.

To see that this is the moment map, note that given an $X \in \mathfrak{sp}_{2n}\mathbb{R}$, the vector field $\underline{X} = (Xv)^t \partial_v$, where ∂_v is shorthand for the obvious column matrix. We further have from direct calculation that $\partial_v \lrcorner\, \omega = -J\mathrm{d}v$, where similarly $\mathrm{d}v$ is the obvious column matrix. Therefore, $\underline{X} \lrcorner\, \omega = -(Xv)^t J\mathrm{d}v$. On the other hand, we also have that

$$\mathrm{d}\mu^X = -\frac{1}{2}\left((X\mathrm{d}v)^t Jv + (Xv)^t J\mathrm{d}v\right).$$

But now

$$(X\mathrm{d}v)^t Jv = ((X\mathrm{d}v)^t Jv)^t = v^t J^t X\mathrm{d}v = v^t(-JX)\mathrm{d}v = v^t X^t J\mathrm{d}v = (Xv)^t J\mathrm{d}v,$$

where in the third equality we have used that $-JX = X^t J$ since $X \in \mathfrak{sp}_{2n}\mathbb{R}$; therefore, the two terms are exactly the same and we get that

$$\mathrm{d}\mu^X = -(Xv)^t J\mathrm{d}v = \underline{X} \lrcorner\, \omega$$

as needed. Finally, we need to check Ad-equivariance, which follows from the computation that if $g \in \mathrm{Sp}_{2n}\mathbb{R}$, and $Y \in \mathfrak{sp}_{2n}\mathbb{R}$ and $v \in \mathbb{R}^{2n}$, then we have that

$$\mu^{\mathrm{Ad}_g Y}(v) = \mu^{gYg^{-1}}(v) = -\frac{1}{2}(gYg^{-1}v)^t Jv = -\frac{1}{2}(Yg^{-1}v)^t g^t Jv = -\frac{1}{2}(Yg^{-1}v)^t Jg^{-1}v = \mu^Y(g^{-1}v),$$

where in the fourth equality we have used that $g^t J = Jg^{-1}$ since $g \in \mathrm{Sp}_{2n}\mathbb{R}$.

Example 6. Consider the standard action of U_n on $(\mathbb{C}^n, \omega_{\mathrm{std}})$; this is in fact a Kähler action almost by definition of U_n, and so certainly symplectic. We use the usual identification of $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ and $\mathrm{GL}_n\mathbb{C} \hookrightarrow \mathrm{GL}_{2n}\mathbb{R}$. Under these identificaitons, we know that $\mathrm{U}_n \hookrightarrow \mathrm{Sp}_{2n}\mathbb{R}$, and so Observation 1 tells us that the action of U_n on $\mathbb{C}^n$ is Hamiltonian with the (co)moment map given by the same formula. There is in fact a better way to express this.

Note that

$$\mathfrak{u}_n = \{X \in \mathfrak{gl}_n\mathbb{C} : X + X^\dagger = 0\} \cong \left\{ \begin{bmatrix} V & -W \\ W & -V \end{bmatrix} \in \mathfrak{gl}_{2n}\mathbb{R} : V + V^t = W - W^t = 0 \right\},$$

with the identification given by writing $X = V + iW$ as usual. Therefore, if we write $v = \begin{bmatrix} x \\ y \end{bmatrix}$, then

$$\mu(X,v) = -\frac{1}{2}\begin{bmatrix} x^t & y^t \end{bmatrix} \begin{bmatrix} 0 & -\mathrm{id}_n \\ \mathrm{id}_n & 0 \end{bmatrix} \begin{bmatrix} V & -W \\ W & -V \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2}(x^t Wx + y^t Wy) + x^t Vy,$$

where we have used that $y^t Vx = (y^t Vx)^t = x^t V^t y = -x^t Vy$. But now if v corresponds to $z = x + iy$, then we also have that

$$-\frac{i}{2}\operatorname{tr}\left(zz^\dagger X\right) = -\frac{i}{2}(z^\dagger Xz) = -\frac{i}{2}(x^t - iy^t)(V + iW)(x + iy) = \frac{1}{2}(x^t Wx + y^t Wy) + x^t Vy,$$

where in the simplification we have used that $x^t Vx = y^t Vy = 0$ because $V = -V^t$. These are identical formulae. Therefore, we have shown that the action of U_n on $(\mathbb{C}^n, \omega_{\mathrm{std}})$ is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{u}_n$ given by $\mu(z) = -\frac{i}{2}zz^\dagger$, where we have implicitly identified $\mathfrak{u}_n$ with its dual using the inner product $\langle A, B\rangle = \operatorname{tr}\left(A^\dagger B\right)$.

Example 7. Now consider the action of U_k on $(\mathbb{C}^{k\times n}, \omega_{\mathrm{std}})$ by left multiplication; this is again a Kähler action by symplectomoprhisms. Then a moment's thought shows that if we use the previous example, then we are in the situation of Observation 2, so this action is Hamiltonian with moment map $\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}_k$ given by $\mu(A) = -\frac{i}{2}AA^\dagger + C$ for any $C \in \mathfrak{u}_k$. As before, we will choose $C = \frac{i}{2}\mathrm{id}_n$, for reasons that will become clearer in Example 9 below.

3 Marsden-Weinstein-Meyer Reduction

In this section, we follow [1], Chapter 23. The idea behind Marsden-Weinstein-Meyer reduction is that in a Hamiltonian G-space, we can under certain hypotheses reduce the degrees of freedom of the system by $2\dim G$ "without affecting the physics". This is captured precisely by the following theorem.

Theorem 2. Let $(M, \omega, G, \rho, \mu)$ be a Hamiltonian G-space for a compact Lie group G. Assume that G acts freely on $\mu^{-1}(0)$. Then

- 0 is a regular value of μ, so that $\mu^{-1}(0) \subset M$ is an embedded submanifold,
- the orbit space $M_{\mathrm{red}} = \mu^{-1}(0)/G$ is a manifold and the map $\pi : \mu^{-1}(0) \to M_{\mathrm{red}}$ a principal G-bundle, and
- there is a symplectic form ω_{red} on M_{red} such that $\omega|_{\mu^{-1}(0)} = \pi^*\omega_{\mathrm{red}}$.

Proof. To prove this, we first need a lemma to understand the differential $\mathrm{d}\mu^X$ a little better.

Lemma 2. Let $(M, \omega, G, \rho, \mu)$ be a Hamiltonian G-space with G compact. Then for any $p \in M$, the stabilizer $G_p \subset G$ is a closed subgroup, and hence an embedded Lie subgroup. If $\mathfrak{g}_p \subset \mathfrak{g}$ is its Lie algebra, then the map $\mathrm{d}\mu_p : T_pM \to \mathfrak{g}^*$ has

$$\ker \mathrm{d}\mu_p = (T_p\mathscr{O}_p)^{\omega_p} \text{ and } \operatorname{im}\mathrm{d}\mu_p = \operatorname{Ann}\mathfrak{g}_p,$$

where $\mathscr{O}_p$ is the orbit of p, which is an embedded submanifold diffeomorphic to G/G_p.

Proof. For all $p \in M$, $v \in T_pM$ and $X \in \mathfrak{g}$, we have that $\omega_p(\underline{X}_p, v) = \langle \mathrm{d}\mu_p(v), X \rangle$. Since $T_p\mathscr{O}_p = \{\underline{X}_p : X \in \mathfrak{g}\}$, it follows immediately that $\ker \mathrm{d}\mu_p \subset (T_p\mathscr{O}_p)^{\omega_p}$. Similarly, if $X \in \mathfrak{g}_p$, then $\underline{X}_p = 0$, so that $\operatorname{im} \mathrm{d}\mu_p \subset \operatorname{Ann} \mathfrak{g}_p$. Therefore, we get that

$$\dim M = \dim T_pM = \dim \ker \mathrm{d}\mu_p + \dim \operatorname{im} \mathrm{d}\mu_p \leq (\dim M - \dim G/G_p) + (\dim \mathfrak{g} - \dim \mathfrak{g}_p) = \dim M,$$

so we must have equality everwhere. ■

It follows that for a given $p \in M$, the following are equivalent:
- the action is locally free at p,
- the stabilizer G_p is discrete,
- the algebra $\mathfrak{g}_p = 0$,
- the differential $\mathrm{d}\mu_p$ is surjective, and
- p is a regular point of μ.

In particular, if G acts freely on $\mu^{-1}(0)$, then 0 is a regular value of μ, and so $\mu^{-1}(0)$ is a closed embedded submanifold of codimension equal to $\dim G$. We also know from elementary differential topology that in this case we have $T_p\mu^{-1}(0) = \ker \mathrm{d}\mu_p$, so that $T_p\mu^{-1}(0)$ and $T_p\mathscr{O}_p$ are symplectic orthogonal complements in T_pM. By Ad-equivariance, it follows that $\mathscr{O}_p \subset \mu^{-1}(0)$, so that $T_p\mathscr{O}_p$ is an isotropic subspace of T_pM. In particular, the orbits in $\mu^{-1}(0)$ are isotropic. Now we need some linear algebra:

Lemma 3. Let (V, ω) be a symplectic vector space and $W \subset V$ isotropic. Then ω induces a caonical symplectic form ω_{red} on W^ω/W such that if $\pi : W^\omega \to W^\omega/W$ is the projection, then $\pi^*\omega_{\text{red}} = \omega|_{W^\omega}$.

Proof. For $[u], [v] \in W^\omega/W$, lift to $u, v \in W^\omega$ and define $\omega_{\text{red}}([u],[v]) := \omega(u, v)$. All properties follow immediately. ■

Finally, from standard differential geometry, since G acts freely and properly on $\mu^{-1}(0)$ and $\mu^{-1}(0)$ is a manifold, we know that $M_{\text{red}} = \mu^{-1}(0)/G$ is a manifold with the map $\pi : \mu^{-1}(0) \to M_{\text{red}}$ a principal G-bundle. In the lemma we take $(V, \omega) = (T_pM, \omega_p)$ and the subspace $W = T_p\mathscr{O}_p$; then $W^\omega = T_p\mu^{-1}(0)$. Since $M_{\text{red}} = \mu^{-1}(0)/G$, it follows immediately that for any $[p] \in M_{\text{red}}$ we have that $T_p\mu^{-1}(0)/T_p\mathscr{O}_p \xrightarrow{\sim} T_{[p]}M_{\text{red}}$. The lemma gives us a canonical symplectic form $\omega_{\text{red},p}$ on $T_{[p]}M_{\text{red}}$, and this is independent of the choice of p because of the G-equivariance of ω. Therefore, we have produced a nondegenerate 2-form ω_{red} on M_{red} that clearly satisfies $\pi^*\omega_{\text{red}} = \omega|_{\mu^{-1}(0)}$ by construction. Now, we have that $\pi^*\mathrm{d}\omega_{\text{red}} = \mathrm{d}\pi^*\omega_{\text{red}} = \mathrm{d}\omega|_{\mu^{-1}(0)} = 0$ and so $\mathrm{d}\omega_{\text{red}} = 0$ because π^* is injective. This completes the proof that $(M_{\text{red}}, \omega_{\text{red}})$ is symplectic. ■

Remark 1. The same proof applies when G is possibly noncompact but acts freely and properly on $\mu^{-1}(0)$; then we have to be only a little more careful, because $G/G_p \xrightarrow{\sim} \mathscr{O}_p \subset M$ is only an injective immersion and not necessarily an embedding. An explanation can be found in [2], Section 5.4.

4 A Couple of Examples

Example 8. Consider the action of S^1 on $(\mathbb{C}^{n+1}, \omega_{\text{std}})$ by left multiplication. As explained in Example 4, this is Hamiltonian with moment map $\mu : \mathbb{C}^n \to \mathfrak{t} \cong \mathbb{R}$ given by $\mu(z) = -\frac{1}{2}|z|^2 + \frac{1}{2}$. Therefore, the preimage $\mu^{-1}(0) = S^{2n+1} \subset \mathbb{C}^{n+1}$. Since S^1 is compact and acts freely on S^{2n+1}, we conclude that the orbit space $M_{\text{red}} = S^{2n+1}/S^1$ is a principal S^1-bundle; this is nothing but $S^1 \to S^{2n+1} \to \mathbb{CP}^n$. The reduced form ω_{red} on $\mathbb{CP}^n$ is called the Fubini-study form ω_{FS}, which also admits several other definitions, notably as the image of $\frac{i}{2}\partial\overline{\partial}\log|z|^2$ on $\mathbb{C}^{n+1}$.

Example 9. Consider the action of U_k on $(\mathbb{C}^{k\times n}, \omega_{\text{std}})$ by left multiplication. In Example 7, we showed that this is Hamiltonian with moment map $\mu : \mathbb{C}^{k\times n} \to \mathfrak{u}_k$ given by $\mu(A) = -\frac{i}{2}AA^\dagger + \frac{i}{2}\,\mathrm{id}_k$ under the identification specified. Therefore, the preimage $\mu^{-1}(0) = \{A \in \mathbb{C}^{k\times n} : AA^\dagger = \mathrm{id}_k\}$. The condition $AA^\dagger = \mathrm{id}_k$ means exactly that the rows of the matrix A are orthonormal with respect to the usual Hermitian inner product on $\mathbb{C}^n$. In particular, A has maximal rank. From this it follows that the action of U_k is free: indeed, in any A, there is a $k \times k$-minor which has full rank, and this minor is witness to an element of $g \in \mathrm{U}_k$ (not) being the identity id_k. Therefore, we are in the situation of the previous theorem. It follows immediately that the quotient $\mu^{-1}(0)/\mathrm{U}_k$ is nothing but the Grassmannian $\mathrm{Gr}_{\mathbb{C}}(k,n)$ of k-planes in $\mathbb{C}^n$; it follows that this has $\dim \mathrm{Gr}_{\mathbb{C}}(k,n) = \dim \mathbb{C}^{k\times n} - 2\dim \mathrm{U}_k = 2kn - 2k^2 = 2k(n-k)$ as expected. In fact, here the action is Kähler, and we can check that this gives $\mathrm{Gr}_{\mathbb{C}}(k,n)$ the structure of not just a symplectic but a Kähler quotient.

References

[1] A. C. da Silva, *Lectures on Symplectic Geometry*. No. 1764 in Lecture Notes in Mathematics, Springer-Verlag Berlin Heidelberg, 2008.

[2] D. McDuff and D. Salamon, *Introduction to Symplectic Topology*. No. 27 in Oxford Graduate Texts in Mathematics, Oxford University Press, third ed., 2016.