Some Remarks on Fundamental Groups and Cohomological Dimension

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Contents

1 Graphs

A *graph* is a CW complex of dimension 1.

Theorem 1.1. Let Γ be a finite connected graph. Then $\pi_1(\Gamma)$ is free on $1 - \chi(\Gamma)$ generators.

Proof. Both $\pi_1(\Gamma)$ and $\chi(\Gamma)$ are homotopy invariant, and any finite connected graph is homotopy equivalent to a wedge of finitely many circles by contracting a maximal tree; then we are done by the Seifert-van Kampen Theorem. ■

Corollary 1.2. Let $n, r \in \mathbb{Z}_{\geq 1}$ and $G \leq F_n$ with $[F_n : G] = r$. Then G is free on $(n-1)r+1$ generators.

Proof. Let X_n be the wedge of n circles, and Γ the covering space corresponding to G. By Theorem 1.1, $G \cong \pi_1(\Gamma)$ is free on $1 - \chi(\Gamma) = 1 - r(1-n)$ generators. ■

2 Fundamental Groups of Manifolds

Theorem 2.1. Let X be a connected closed Kähler manifold which is not simply connected. Then $\pi_1(X)$ is not a free group.

Proof. Suppose X is a connected closed Kähler manifold. Since X is Kähler, the first Betti number $b_1(X)$ is even, say $b_1(X) = 2g$ for $g \in \mathbb{Z}_{\geq 0}$. If $\pi_1(X)$ is free, then by considering its abelianization it is free of rank $n = 2g$. If X is not simply connected, then $g \geq 1$. In particular, $\pi_1(X)$ has a subgroup G of index 2. By Corollary 1.2, G is free of rank $2n - 1$. Then the corresponding double cover $\widetilde{X}$ of X is also a connected closed Kähler manifold with $b_1(\widetilde{X}) = 2n - 1$, which is a contradiction. ■

Theorem 2.2. Let $n \in \mathbb{Z}_{\geq 1}$ be an integer, X be a connected complete Riemannian n-manifold of everywhere non-positive sectional curvature. Let $G := \pi_1(X)$.

(a) For any ring R, the R-cohomological dimension $\mathrm{cd}_R(G) \leq n$. If X is not closed, then $\mathrm{cd}_R(G) \leq n-1$.
(b) If X is closed (resp. closed orientable), then for any $\mathbb{F}_2$-algebra R (resp. any ring R) we have $\mathrm{cd}_R(G) = n$. In particular, if $n \geq 2$, then G is not free.

Proof. Let $G := \pi_1(X)$. By the Cartan-Hadamard Theorem, X is a $K(G,1)$, and hence for any ring R we have $\mathrm{cd}_R(G) \leq n$. If X is not closed, it is homotopy equivalent to a CW complex of dimension $n-1$ or lower. In the setting of (b), we have by Poincaré duality that $\mathrm{H}^n(X,R) = R \neq 0$; in particular, the trivial $R[G]$-module R demonstrates that $\mathrm{cd}_R(G) \geq n$. ■

As a corollary of either of these theorems, we get that if $g \geq 1$ and Σ_g is the surface of genus g, then $\pi_1(\Sigma_g)$ is not free. In fact, for any ring R we have $\mathrm{cd}_R(\pi_1(\Sigma_g)) = 2$.

3 Knot Complements

Proposition 3.1. The group $G = \langle x, y | y^2 x^{-3} \rangle$ has $\mathrm{cd}(G) = 2$.

Proof. This group is the fundamental group of the trefoil knot complement, which is connected, complete, non-closed hyperbolic 3-manifold; therefore, by Theorem 2.2(a), we have $\mathrm{cd}(G) \leq 2$.

By the Stallings-Swan Theorem, it suffices to prove that G is not free. If it were, then since $G^{\mathrm{ab}} \cong \mathbb{Z}$, it would follow that $G \cong \mathbb{Z}$, which it is not since it admits a surjection to S_3 (famously via tricolorings or considering simply $x \mapsto (123), y \mapsto (12)$). ■

In fact, we have

Theorem 3.2. Let $K \subset S^3$ be a smooth nontrivial hyperbolic knot (i.e. such that the complement $S^3 \smallsetminus K$ admits a complete hyperbolic Riemannian metric). Then $\mathrm{cd}(\pi_1(S^3 \smallsetminus K)) = 2$.

Proposition 3.1 is the special case where K is the trefoil knot. This proposition also applies to 4_1 (the figure-eight knot), 5_2 (the three-twist knot), 6_1 (the stevedore knot), etc.

Proof. By the same argument as in Proposition 3.1, noting that $\pi_1(S^3 \smallsetminus K)$ is not isomorphic to $\mathbb{Z}$ since K is not the unknot. ■