Covariant Derivatives, Connections, and Curvature

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Abstract

This paper is written in fulfillment of the requirements of Math 230A taught at Harvard in Fall 2021 by Dr. Martin Lesourd. It covers the fundamentals of covariant derivatives (or equivalently Koszul connections) on smooth vector bundles, and their curvature, with an introduction to bundle-valued differential forms. It ends with the Gauss Curvature Equation and two concrete computations: for the round sphere and the torus in $\mathbb{R}^3$ with minor radius 1 and major radius 2.

Contents

1 Introduction

1.1 Bundle-Valued Forms

This section closely follows §21 of [1] and Chapter 9 of [2].

Definition 1. Let M be a smooth manifold and $\pi : E \to M$ be a smooth vector bundle. For any integer $q \geq 0$, an *E-valued q-form on M* is a section of the vector bundle $\Lambda^q T^*M \otimes E$. The space of E-valued q-forms on M is denoted by $\Omega^q(M;E) := \Gamma(\Lambda^q T^*M \otimes E)$. The total vector space of all E-valued forms on M is denoted $\Omega^*(M;E) = \bigoplus_{q \geq 0} \Omega^q(M;E)$.

We can write these explicitly in local coordinates: if $\{\mathrm{e}_i\}$ is a local frame for E, then a typical E-valued q-form in these coordinates looks like $\alpha^i \mathrm{e}_i$, where α^i are ordinary differential q-forms and we are suppressing the tensor product symbol.

Example 1. When $E = M \times V$ is a trivial vector bundle with fiber V, then we call an E-valued form a *V-valued form* on a vector-valued form with values in E. The space of V-valued q-forms on E is often written $\Omega^q(M;V)$. The following are some familiar examples.

(a) When $V = \mathbb{R}$, then we get simply the differential forms, i.e. $\Omega^q(M;\mathbb{R}) = \Omega^q(M)$.
(b) When $V = \mathbb{R}^n$, then we get column vectors of differential forms. Similarly, when $V = \mathbb{R}^{m \times n}$ is the space of $m \times n$ matrices, then we get the matrix-valued differential forms.
(c) More generally, if $V = \mathfrak{g}$ is a Lie algebra, then we get the Lie-algebra valued differential forms $\Omega^q(M;\mathfrak{g})$.

Example 2. Here are a couple of examples of TM-valued forms.

(a) Corresponding to $1_{TM} \in \mathrm{Hom}_{\mathbb{R}}(TM, TM) \cong T^*M \otimes TM$ we get the *identity form* $\iota \in \Omega^1(M;TM)$.
(b) Let $X \in \Gamma(TM)$ be a vector field on the manifold M. Then the Lie derivative along X gives us $\mathscr{L}_X \in \mathrm{Hom}_{\mathbb{R}}(TM, TM) \cong T^*M \otimes TM$ and hence a TM-valued 1-form $\mathscr{L}_X \in \Omega^1(M;TM)$.

Example 3. Let $f : M \to N$ be a smooth map. Then its differntial $\mathrm{d}f : TM \to f^*TN$ is an element of $\mathrm{Hom}_{\mathbb{R}}(TM, f^*TN)$ and hence gives us an f^*TN-valued 1-form $\mathrm{d}f \in \Omega^1(M; f^*TN)$. When $N = \mathbb{R}$, then $TN \cong \mathbb{R} \times \mathbb{R}$ and $f^*TN \cong M \times \mathbb{R}$; under these identifications, the derivative

$$\mathrm{d}f \in \Omega^1(M; f^*TN) = \Omega^1(M;\mathbb{R}) = \Omega^1(M)$$

is simply the usual exterior derivative.

We will see below that given a covariant derivative ∇ on a vector bundle E, the curvature of E belongs to $\Omega^2(M; \mathrm{End}_{\mathbb{R}}(E))$, i.e. is an $\mathrm{End}_{\mathbb{R}}(E)$-valued 2-form on M.

1.2 Products of Bundle-Valued Forms

Let E, F, G be smooth vector bundles over the smooth base M. We consider bundle maps $\mu : E \otimes F \to G$ i.e. elements $\mu \in \Gamma((E \otimes F)^* \otimes G)$.

Example 4. IF E is an vector bundle, then the wedge product gives rise to a bundle map $\wedge : \Lambda^p E \otimes \Lambda^q E \to \Lambda^{p+q} E$, i.e. an element of $(\Lambda^p E \otimes \Lambda^q E)^* \otimes \Lambda^{p+q} E$. In particular, taking $E = T^*M$ gives the usual wedge product of differential forms.

Given two such bundle maps $\mu : E \otimes F \to G$ in $\Gamma((E \otimes F)^* \otimes G)$ and $\mu' : E' \otimes F' \to G'$ in $\Gamma((E' \otimes F')^* \otimes G')$, we can produce a new one by taking their tensor product to get

$$\mu \otimes \nu : (E \otimes E') \otimes (F \otimes F') \to G \otimes G'$$

given by $\mu \otimes \nu \in \Gamma((E \otimes F)^* \otimes G \otimes (E' \otimes F')^* \otimes G) \cong \Gamma((E \otimes E' \otimes F \otimes F')^* \otimes G \otimes G')$. In particular, combining this with the previous example and taking global sections, we see that given such a map $\mu : E \otimes F \to G$, we get a bundle map

$$\wedge \otimes \mu : \Omega^p(M;E) \times \Omega^q(M;F) \to \Omega^{p+q}(M;G)$$

given in local coordinates by $(\alpha^i \mathrm{e}_i, \beta^j \mathrm{f}_j) \mapsto (\alpha^i \wedge \beta^j)\mu(\mathrm{e}_i, \mathrm{f}_j)$.

Example 5. Take E, F, G to be trivial bundles with fibers U, V, W respectively, and let $\mu : U \times V \to W$ be a bilinear map. Then we also of course get a bundle map $\mu : E \otimes F \to G$, and hence a map

$$\wedge \otimes \mu : \Omega^p(M;U) \times \Omega^q(M;V) \to \Omega^{p+q}(M;W).$$

This map is occasionaly simply denoted using $\cdot$ for convenience, but it is important to note that it depends on μ.

(a) We can take $U = \mathbb{R}^{m\times n}$, $V = \mathbb{R}^{n\times k}$, $W = \mathbb{R}^{m\times k}$, and μ to be matrix multiplication. Then this operation $\wedge \otimes \mu$ gives the operation of taking the wedge product of a matrix of forms (which can be used to succicintly state the structural identities, for instance), which we write simply as $(\alpha, \beta) \mapsto \alpha \wedge \beta$.

(b) We can take $U = V = W = \mathfrak{g}$ to be a Lie algebra, and $\mu = [,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ to be the Lie bracket. Then we get a Lie bracket of $\mathfrak{g}$-valued forms, which we write simply as $(\alpha, \beta) \mapsto [\alpha, \beta]$. This can be expressed most conveniently in a basis: if $v_1, \ldots, v_n \in \mathfrak{g}$ is any set of vectors and $\alpha = \alpha^i v_i \in \Omega^p(M;\mathfrak{g})$ and $\beta = \beta^j v_j \in \Omega^q(M;\mathfrak{g})$, then $[\alpha, \beta] = (\alpha^i \wedge \beta^j)[v_i, v_j] \in \Omega^{p+q}(M;\mathfrak{g})$. From this we get the graded skew-symmetry property: if $\alpha \in \Omega^p(M;\mathfrak{g})$ and $\beta \in \Omega^q(M;\mathfrak{g})$, then $[\beta, \alpha] = (-1)^{pq+1}[\alpha, \beta]$, and indeed the proof is one-linear. Take any basis $v_1, \ldots, v_n$ of $\mathfrak{g}$ and expand $\alpha = \alpha^i v_i$ and $\beta = \beta^j v_j$. Then

$$[\beta, \alpha] = (\beta^j \wedge \alpha^i)[v_j, v_i] = (-1)^{pq}(\alpha^i \wedge \beta^j) \cdot (-[v_i, v_j]) = (-1)^{pq+1}\alpha^i \wedge \beta^j [v_i, v_j] = (-1)^{pq+1}[\alpha, \beta].$$

In the case of $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) \cong \mathbb{R}^{n\times n}$, we get two operations $\wedge$ and $[,]$ on $\mathfrak{g}$-valued forms on a manifold. The commutator relation $[A, B] = AB - BA$, then immediately implies that if $\alpha \in \Omega^p(M;\mathfrak{gl}(n, \mathbb{R}))$ and $\beta \in \Omega^q(M;\mathfrak{gl}(n;\mathbb{R}))$, then $[\alpha, \beta] = \alpha \wedge \beta - (-1)^{pq}\beta \wedge \alpha \in \Omega^{p+q}(M;\mathfrak{gl}(n;\mathbb{R}))$.

Example 6. Let E be any vector bundle. Then scalar multiplication gives a bundle map[1] $\mu : \mathbb{R} \otimes E \to E$, and hence we get a product map $\wedge \otimes \mu : \Omega^p(M) \times \Omega^q(M;E) \to \Omega^{p+q}(M;E)$ which we write simply as $(\omega, \sigma) \mapsto \omega \wedge \sigma$, where $\omega \in \Omega^p(M)$ is an ordinary p-form and $\sigma \in \Omega^q(M;E)$ is an E-valued q-form. For instance, when $p = q = 0$, then we simply get the scaling map $C^\infty(M) \times \Gamma(E) \to \Gamma(E)$ given by $(f, \sigma) \mapsto f\sigma$.

Example 7. Let E be any vector bundle. Then evaluation gives a bundle map $\mathrm{ev} : \mathrm{End}_{\mathbb{R}}(E) \otimes E \to E$. Applying the above construction gives us a map

$$\wedge \otimes \mathrm{ev} : \Omega^p(M;\mathrm{End}_{\mathbb{R}} E) \times \Omega^q(M;E) \to \Omega^{p+q}(M;E)$$

given in local coordinates by $(\alpha^i \varphi_i, \beta^j \sigma_j) \mapsto \alpha^i \wedge \beta^j \varphi_i(\sigma_j)$. This is also typically denoted simply using $\wedge$.

[1] Here $\mathbb{R}$ denotes the trivial bundle $M \times \mathbb{R}$.

2 Covariant Derivative

This section again closely follows [1] and [2].

2.1 Definition and Construction

Definition 2. Let $\pi : E \to M$ be a smooth vector bundle. Then *a covariant derivative on E* is an $\mathbb{R}$-linear map $\nabla : \Omega^*(M;E) \to \Omega^{*+1}(M;E)$ which satisfies the Leibniz rule, i.e. such that for all $\omega \in \Omega^p(M)$ and $\sigma \in \Omega^q(M;E)$ we have that

$$\nabla(\omega \wedge \sigma) = \mathrm{d}\omega \wedge \sigma + (-1)^p \omega \wedge \nabla\sigma \in \Omega^{p+q+1}(M;E).$$

To construct such things, we often employ the locality principle.

Lemma 1 (Locality Principle)**.** If ∇ is a covariant derivative on a vector bundle $E \to M$, then if a form $\sigma \in \Omega^q(M;E)$ vanishes identically in a neighborhood V of a point $x \in M$, then $(\nabla\sigma)_x = 0$.

Proof. Let f be a smooth bump function on M such that $f \equiv 1$ on a neighborhood of x and $\operatorname{supp} f \subset V$. Then $f\sigma = 0$, so that $0 = \nabla(f\sigma) = \mathrm{d}f \wedge \sigma + f\nabla\sigma$. Evaluating at x and using $f(x) = 1$ and $\mathrm{d}f_x = 0$ gives us the result. ■

Theorem 1 (Restriction Theorem)**.** If ∇ is a covariant derivative on $E \to M$, then for any open $U \subseteq M$, there is a unique covariant derivative $\nabla|_U$ on $E|_U$ such that for all $\sigma \in \Omega^*(M;E)$ we have $\nabla|_U \sigma|_U = \nabla\sigma|_U$.

Proof. Let $\sigma \in \Omega^p(U;E|_U)$ and $x \in U$ be given. By scaling by an appropriate bump function, we can pick a global $\widetilde{\sigma} \in \Omega^p(M;E)$ that agrees with σ in a neighborhood of x; then we define $(\nabla|_U \sigma)_x := (\nabla\widetilde{\sigma})_x$. By the Locality Principle, this is independent of the choice of extension $\widetilde{\sigma}$. That it is smooth, $\mathbb{R}$-linear, and satisfies the Leibniz rule follows immediately, and the proof is left as an exercise. The last condition and uniqueness is clear by construction. ■

Theorem 2 (Construction Theorem)**.** If $E \to M$ is a vector bundle, $\mathfrak{U}$ an open cover of M, and $\{\nabla_U\}_{U \in \mathfrak{U}}$ a family of covariant derivatives on the various $E|_U$ such that for all $U, U' \in \mathfrak{U}$ we have $\nabla_U|_{U \cap U'} = \nabla_{U'}|_{U \cap U'}$, then there is a unique covariant derivative ∇ on E such that for every $U \in \mathfrak{U}$ we have $\nabla|_U = \nabla_U$.

Proof. Given a global $\sigma \in \Omega^p(M;E)$ and $x \in M$, pick a $U \in \mathfrak{U}$ containing x and define $(\nabla\sigma)_x := (\nabla_U \sigma|_U)_x \in \Lambda^{p+1} T_x^* M \otimes E_x$. By the compatibility condition, this is independent of the $U \in \mathfrak{U}$ containing x that is chosen. Again, smoothness, $\mathbb{R}$-linearly, and the Leibniz rule follow immediately by the same argument as above, and the proof is left as an exercise. The last condition and uniqueness is clear by construction. ■

The Construction Theorem shows that the space of connections on a vector bundle forms a sheaf. It tells us that to construct a covariant derivative, it suffices to construct the derivative on a framed open set, as long as the formula you use transforms correctly under a change of frame. In fact, we don't even need to construct ∇ at each level: simply constructing at $* = 0$ suffices. This explains the equivalence with the more traditional definition.

Theorem 3. Let $\pi : E \to M$ be a smooth vector bundle. Let $\nabla_0 : \Gamma(E) \to \Gamma(T^*M \otimes E)$ be an $\mathbb{R}$-linear map which satisfies the Leibniz rule, i.e. such that for $f \in \mathscr{C}^\infty(M)$ and $\sigma \in \Gamma(E)$ we have

$$\nabla_0(f\sigma) = \mathrm{d}f \otimes \sigma + f\nabla_0\sigma.$$

Then there is a unique covariant derivative $\nabla : \Omega^*(M;E) \to \Omega^{*+1}(M;E)$ such that ∇ agrees with ∇_0 at the $* = 0$ level.

Proof. In light of the Construction Theorem, it suffices to assume that E admits a global frame $\mathrm{e}_1, \ldots, \mathrm{e}_n$, so every element $\sigma \in \Omega^q(M;E)$ looks like $\sigma = \sigma^i \mathrm{e}_i$ for some $\sigma^i \in \Omega^q(M)$. If such a ∇ is to exist, then it must satisfy

$$\nabla\sigma = \nabla(\sigma^i \mathrm{e}_i) = \nabla(\sigma^i \wedge \mathrm{e}_i) = \mathrm{d}\sigma^i \mathrm{e}_i + (-1)^q \sigma^i \wedge \nabla_0 \mathrm{e}_i,$$

which shows uniqueness if such a formula works. To show that it does indeed, we need to check that with this definition ∇ is $\mathbb{R}$-linear (easy), that it satisfies the Leibniz rule, and that it is independent of the choice of frame.

For the Leibniz rule, observe that if $\omega \in \Omega^p(M)$ and $\sigma \in \Omega^q(M;E)$ then if $\sigma = \sigma^i \mathrm{e}_i$, then $\omega \wedge \sigma = (\omega \wedge \sigma^i)\mathrm{e}_i$. Therefore,

$$\nabla(\omega \wedge \sigma) = \nabla((\omega \wedge \sigma^i)\mathrm{e}_i) = \mathrm{d}\big(\omega \wedge \sigma^i\big)\,\mathrm{e}_i + (-1)^{p+q}(\omega \wedge \sigma^i) \wedge \nabla_0 \mathrm{e}_i.$$

Now $\mathrm{d}\big(\omega \wedge \sigma^i\big) = \mathrm{d}\omega \wedge \sigma^i + (-1)^p \omega \wedge \mathrm{d}\sigma^i$, so that this becomes

$$\mathrm{d}\omega \wedge \sigma^i \mathrm{e}_i + (-1)^p \omega \wedge \mathrm{d}\sigma^i \mathrm{e}_i + (-1)^{p+q}(\omega \wedge \sigma^i) \wedge \nabla_0 \mathrm{e}_i = \mathrm{d}\omega \wedge \sigma + (-1)^p \omega \wedge \nabla\sigma,$$

where in the last step we've used the definition of $\nabla\sigma$ and associtivity $(\omega \wedge \sigma^i) \wedge \nabla_0 \mathrm{e}_i = \omega \wedge (\sigma^i \wedge \nabla_0 \mathrm{e}_i)$. Finally, we need to check that this is independent of the local frame chosen, so suppose that $\mathrm{f}_1, \dots, \mathrm{f}_n$ is another local frame, so that there are smooth functions g^i_j such that $\mathrm{e}_j = g^i_j \mathrm{f}_i$ and the matrix $[g^i_j]$ is everywhere invertible. Then $\sigma = \sigma^j \mathrm{e}_j = (\sigma^j g^i_j)\mathrm{f}_i$, so computing in f_i gives

$$\nabla\sigma = \mathrm{d}\big(\sigma^j g^i_j\big)\,\mathrm{f}_i + (-1)^p \sigma^j g^i_j \wedge \nabla_0 f_j = (\mathrm{d}\sigma^j) g^i_j \mathrm{f}_i + (-1)^p \sigma^j \wedge (\mathrm{d}g^i_j)\mathrm{f}_i + (-1)^p \sigma^j g^i_j \wedge \nabla_0 \mathrm{f}_j.$$

Since $\nabla_0 \mathrm{e}_j = \nabla_0(g^i_j \mathrm{f}_i) = (\mathrm{d}g^i_j)\mathrm{f}_i + g^i_j \nabla_0 \mathrm{f}_j$, this expression above is exactly $\mathrm{d}\sigma^j \mathrm{e}_j + (-1)^p \sigma^j \wedge \nabla_0 \mathrm{e}_j$. ■

Example 8. Let E be the trivial vector bundle with fiber V. By sending all the constant sections $v \in V \subset \Gamma(E)$ to 0, we can define a $\nabla_0 : \Gamma(E) \to \Gamma(T^*M \otimes E)$ by $\nabla_0(f^i v_i) = (\mathrm{d}f^i)v_i$. This is clearly $\mathbb{R}$-linear and satisfies the Leibniz rule, and hence extends to an exterior derivative $\nabla : \Omega^*(M;V) \to \Omega^{*+1}(M;V)$ which is just the componentwise exterior derivative and hence also simply denoted using d. This provides a way of differentiating vector valued, matrix valued, and Lie-algebra-valued differential forms. In this direction, we get that this exterior derivative of vector-valued forms is an antiderivation in the following sense: if U, V, W are vector spaces and $\mu : U \times V \to W$ a bilinear map, then for $\alpha \in \Omega^p(M;U)$ and $\beta \in \Omega^q(M;V)$ we have that

$$\mathrm{d}(\alpha \cdot \beta) = (\mathrm{d}\alpha) \cdot \beta + (-1)^p \alpha \cdot \mathrm{d}\beta,$$

a proposition that is evident from the antiderivation property for forms when everything is expanded out in a basis.

2.2 Covariant Derivatives from Koszul Connections

Definition 3. A *Koszul connection* on a smooth vector bundle $E \to M$ is a map $\overline{\nabla} : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ written $(X, \sigma) \mapsto \overline{\nabla}_X \sigma$ that is $\mathscr{C}^\infty(M)$-linear in X, $\mathbb{R}$-linear in $\Gamma(E)$, and satisfies the Leibniz rule

$$\overline{\nabla}_X(f\sigma) = (Xf)\sigma + f\overline{\nabla}_X \sigma$$

for $f \in \mathscr{C}^\infty(M)$.

Given a Koszul connection $\overline{\nabla}$, we get an associated covariant derivative

$$\nabla : \Gamma(E) \to \Gamma(T^*M \otimes E) \cong \mathrm{Hom}_{\mathscr{C}^\infty(M)}(\Gamma(TM), \Gamma(E))$$

taking $\sigma \mapsto (X \mapsto \overline{\nabla}_X \sigma)$, which can then be extended by Theorem 3. Conversely, given a covariant derivative ∇, we get a Koszul conenction $\nabla : \Gamma(TM) \times \Gamma(E) \to \Gamma(E)$ by $(X, \sigma) \mapsto (\nabla\sigma)(X)$, i.e. given by tensoring with TM and taking the trace or contracting. Since both of these latter operations are $\mathscr{C}^\infty(M)$-linear, this is $\mathscr{C}^\infty(M)$ linear in X. The other properties are also an immediate consequence of the observation $(\mathrm{d}f)(X) := Xf$. Therefore, the notions of a Koszul connection and of a covariant derivative on a vector bundle are identical. From this point onwards, we will use ∇ for both the covariant derivative and Koszul connection interchangably.

Definition 4. An *affine connection* on a manifold M means a Koszul connection (or equivalently covariant derivative) on the tangent bundle TM.

Example 9. When $M = \mathbb{R}^n$, we get an affine connection $\overline{\nabla}^{\mathrm{Euc}} : \Gamma(T\mathbb{R}^n) \times \Gamma(T\mathbb{R}^n) \to \Gamma(T\mathbb{R}^n)$ given by

$$(X^i \partial_i, Y^j \partial_j) \mapsto X^i \partial_i(Y^j)\partial_j.$$

This is called the *Euclidean connection*. The associated covariant derivative is then given simply by $Y^j \partial_j \mapsto \partial_i Y^j \mathrm{d}x^i \otimes \partial_j$. More generally, let $M \subset \mathbb{R}^n$ be an embedded submanifold. Then we define an affine connection on M by $\overline{\nabla}_X Y := \mathrm{pr}_{TM}(\overline{\nabla}^{\mathrm{Euc}}_{\widetilde{X}}(\widetilde{Y}))$, where $\widetilde{X}$ and $\widetilde{Y}$ are any (local) smooth extensions. This is clearly independent of the choice of $\widetilde{X}$, and to show that it is independent of the choice of $\widetilde{Y}$, which follows from the characterisation of $T_x M := \{X \in T_x\mathbb{R}^n : Xf = 0 \text{ when } f|_M = 0\}$. This is called the *Riemannian connection* on M.

2.3 Connections on a Framed Open Set

As promised, we look at explicit formulae for connections on framed open sets. Let $E \to M$ be a vector bundle and $\{e_1, \ldots, e_n\}$ be a local frame for E along U. Then any covariant derivative ∇ on E along U gives $\nabla e_j = \omega^i_j e_i$ for some 1-forms $\omega^i_j \in \Omega^1(U)$. If we let e denote the row vector $(e_1, \ldots, e_n)$ and ω the matrix $[\omega^i_j]$, then this can be written succinctly as $\nabla e = e\omega$.

Definition 5. The 1-forms ω^i_j are called the *connection forms* of ∇ with respect to this framed set, and the matrix $\omega := [\omega^i_j]$ of 1-forms is called the *connection matrix* of ∇ with respect to this open set.

Clearly, the covariant derivative $\nabla|_U$ is determined completely by its connection matrix, since we must have $\nabla(g^j e_j) = (dg^i + g^j \omega^i_j)e_i$. Conversely, given any matrix $\omega = [\omega^i_j]$ of 1-forms on U, we can construct a covariant derivative on U by defining ∇ by this formula; that this defines a covariant derivative is an easy check. Therefore, to use this strategy to produce a global covariant derivative, we must ask how the connection matrix transforms under change of local frame.

Therefore, let $\bar{e}$ be a different local frame of E, so that $\bar{e} = ea$ for an $a \in \mathscr{C}^\infty(U, \mathrm{GL}(n, \mathbb{R}))$. Then

$$\nabla\bar{e} = \nabla(ea) = (\nabla e)a + e\,da = e\omega a + e da = \bar{e}(a^{-1}\omega a + a^{-1}da),$$

so the transformed matrix $\overline{\omega} = a^{-1}\omega a + a^{-1}da$.

Example 10. Let ∇ be an affine connection on a manifold M^n. Then in terms of local coordinates $\{x^i\}$ on M, we get a local frame for TM given by the $\{\partial_i\}$ and for T^*M by the dx^i. Then writing $\nabla\partial_j = \omega^i_j\partial_i$ and then $\omega^i_j = \Gamma^i_{jk}dx^k$ we get n^3 smooth functions $\{\Gamma^i_{jk}\}$ called the Christoffel symbols of the affine connection in these local coordinates. The connection matrix in terms of the Christoffel symbols is given by $[\Gamma^i_{jk}dx^k]$. For the Levi-Civita connection on a Riemmannian M, there is a neat formula for the Γ^i_{jk} in terms of the metric g:

$$\Gamma^i_{jk} = \frac{1}{2}g^{i\ell}(\partial_j g_{\ell k} + \partial_k g_{\ell k} - \partial_\ell g_{jk}).$$

A proof of this can be found in [3] or [4] Chapter 5.

2.4 Metric Connections

Let $\pi : E \to M$ be a Riemannian bundle, i.e. a vector bundle with a Riemannian metric, which is by definition an everywhere nondegenerate symmetric 2-covector field $\langle\cdot,\cdot\rangle \in \Gamma(\Sigma^2 E^*)$.

Definition 6. Let ∇ be a connection on a Riemannian bundle E. Then ∇ is said to be a *metric connection* or compatible with the metric if for all $\sigma, \tau \in \Gamma(E)$ and $X \in \Gamma(TM)$ we have that

$$X\langle\sigma,\tau\rangle = \langle\nabla_X\sigma,\tau\rangle + \langle\sigma,\nabla_X\tau\rangle.$$

The key result here is the following.

Lemma 2. Let $E \to M$ be a Riemannian bundle and ∇ a connection on E. If ∇ is compatible with the metric, then its connection matrix $[\omega^i_j]$ relative to any orthornomal fram e for E over a trivializing open set $U \subset M$ is skew-symmetric.

Proof. For all $X \in \Gamma(TU)$ and i, j we have

$$0 = X\langle e_i, e_j\rangle = \langle\nabla_X e_i, e_j\rangle + \langle e_i, \nabla_X e_j\rangle = \langle\omega^k_i(X)e_k, e_j\rangle + \langle e_i, \omega^k_j(X)e_k\rangle = \omega^j_i(X) + \omega^i_j(X).$$

■

Example 11. For instance, the Euclidean connection ∇^{Euc} on $\mathbb{R}^n$ is compatible with the metric, as is easily seen, and its connection matrix with respect to the standard basis $\{\partial_i\}$ is identically 0. More generally, the same result shows that if $M \subset \mathbb{R}^n$ is an embedded submanifold, then ∇^{Euc} on $T\mathbb{R}^n|_M$ is still compatible with the metric.

3 Curvature

This section follows the treatment in [3] and [2]. The treatment of the Gauss Curvature Equation follows [1] §12. The computation in the examples is original.

3.1 Introduction to Curvature

The exterior derivative d satisfies $\mathrm{d}^2 = 0$. However, the exterior covariant derivative on a vector bundle need not square to zero. In fact, its failure to be a cochain differential is measured exactly by the *curvature* of the vector bundle.

Lemma 3. Let ∇ be a covariant derivative on a smooth vector bundle $E \to M$. Then there is a unique element $F_\nabla \in \Omega^2(M;\mathrm{End}_{\mathbb{R}} E)$ such that for all $\sigma \in \Omega^*(M;E)$ we have that $\nabla^2\sigma = F_\nabla \wedge \sigma$, where the $\wedge$ here is shorthand for $\wedge \otimes \mathrm{ev}$ (see Example 7).

Proof. It suffices to proceed locally, so let $\mathrm{e}_- = (\mathrm{e}_i)$ be a local frame and let $\mathrm{e}^- = (\mathrm{e}^i)$ be the corresponding dual frame. Write $\Omega^q(M;E) \ni \sigma = \alpha^i \mathrm{e}_i$ for some $\alpha^i \in \Omega^q(M)$. Then $\nabla\sigma = (\mathrm{d}\alpha^i + (-1)^q(\alpha^j \wedge \omega^i_j))\mathrm{e}_i$. Applying this formula to itself, one line of calculation yields

$$\nabla^2\sigma = (\mathrm{d}\omega^i_j + \omega^i_j \wedge \omega^k_j) \wedge \alpha^j \mathrm{e}_i = F_\nabla \wedge \sigma$$

where $F_\nabla := (\mathrm{d}\omega^i_j + \omega^i_k \wedge \omega^k_j)\mathrm{e}^j \otimes \mathrm{e}_i$. That this transforms correctly using the transformation formula above to yield a global element of $\Omega^2(M;\mathrm{End}_{\mathbb{R}} E)$ is left to the reader as a moderately tedious exercise. ∎

Definition 7. The element $F_\nabla \in \Omega^2(M;\mathrm{End}_{\mathbb{R}} E)$ is called the *curvature* of the covariant derivative ∇. When written in components in a local frame $\{\mathrm{e}_i\}_i$ as $F_\nabla = \Omega^i_j \mathrm{e}^j \otimes \mathrm{e}_i$, the coefficients Ω^i_j are called the *curvature 2-forms* of ∇ with respect to this framed open set. The matrix $\Omega := [\Omega^i_j]$ of 2-forms is called the *curvature matrix* of ∇ with respect to this framed open set.

In the course of proving the above result, we have also shown

Theorem 4 (Second Structural Equation)**.** The curvature matrix Ω and the connection matrix ω satisfy

$$\Omega = \mathrm{d}\omega + \omega \wedge \omega.$$

To find the transformation law, write $\bar{\mathrm{e}} = \mathrm{e}a$ as before, so that $\bar{\mathrm{e}}_j = a^i_j \mathrm{e}_i$ so $\mathrm{e}_j = (a^{-1})^i_j \bar{\mathrm{e}}_i$ and $\mathrm{e}^j = a^j_k \bar{\mathrm{e}}^k$. Then

$$\Omega^i_j \mathrm{e}^j \otimes \mathrm{e}_i = \Omega^i_j a^j_k \bar{\mathrm{e}}^k \otimes (a^{-1})^\ell_i \bar{\mathrm{e}}_\ell = ((a^{-1})^\ell_i \Omega^i_j a^j_k)\bar{\mathrm{e}}^k \otimes \bar{\mathrm{e}}_\ell = \overline{\Omega}^\ell_k \bar{\mathrm{e}}^k \otimes \bar{\mathrm{e}}_\ell,$$

so that $\overline{\Omega} = a^{-1}\Omega a$ as needed.

The curvature calculation can also be expressed in terms of the associated Koszul connection as follows. We use the following two formulae whose proofs are immediate.

Lemma 4.

(a) If $\sigma, \tau \in \Omega^1(M)$, then for $X, Y \in \Gamma(TM)$ we have $(\sigma \wedge \tau)(X,Y) = \sigma(X)\tau(Y) - \sigma(Y)\tau(X)$.
(b) If $\sigma \in \Omega^1(M)$ and $X, Y \in \Gamma(TM)$, then $(\mathrm{d}\sigma)(X,Y) = X\sigma(Y) - Y\sigma(X) - \sigma[X,Y]$.

Using these formulae, expansion in terms of the connection forms $[\omega^i_j]$ it follows immediately that if $\alpha \in \Omega^1(M;E)$, then for any $X, Y \in \Gamma(TM)$ we have that

$$(\nabla\alpha)(X,Y) = \nabla_X(\alpha(Y)) - \nabla_Y(\alpha(X)) - \alpha[X,Y].$$

Therefore, if we have $X, Y \in \Gamma(TM)$ and $\sigma \in \Gamma(E)$, then to compute

$$F_\nabla(X,Y)(\sigma) = (F_\nabla \wedge \sigma)(X,Y) = \nabla^2\sigma(X,Y) = \nabla_X(\nabla\sigma(Y)) - \nabla_Y(\nabla\sigma(X)) - \nabla\sigma[X,Y]$$

so that using $\nabla\sigma(Z) = \nabla_Z\sigma$ by definition, we conclude that

$$F_\nabla(X,Y) = \nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]}.$$

3.2 Gauss Curvature Equation

Let $M \subset \mathbb{R}^3$ be a smooth surface, and let e_1, e_2 be an orthonormal frame for TM on a subset $U \subset M$. If we define $e_3 := e_1 \times e_2$, then e_1, e_2, e_3 is an orthonormal frame for $T\mathbb{R}^3|_M$ over U. For the Euclidean connection ∇^{Euc} on $T\mathbb{R}^3|_M$, let $\omega^{\text{Euc}} = [\omega^i_j]$ denote the connection matrix. From Example 11, the matrix ω^{Euc} is skew-symmetric and hence looks like

$$\omega^{\text{Euc}} = \begin{bmatrix} 0 & \omega^1_2 & \omega^1_3 \\ -\omega^1_2 & 0 & \omega^2_3 \\ -\omega^1_3 & -\omega^2_3 & 0 \end{bmatrix}.$$

By Example 9, the Riemannian connnection ∇ on M is then given simply by the connection matrix

$$\omega = \begin{bmatrix} 0 & \omega^1_2 \\ -\omega^1_2 & 0 \end{bmatrix}.$$

Using the Second Structural Equation (Theorem 4), the curvature matrix Ω of M can be calculated as

$$\Omega = d\omega + \omega \wedge \omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} d\omega^1_2,$$

so the curvature matrix is completely described by $\Omega^1_2 = d\omega^1_2$. Since the Gaussian curvature of ∇^{Euc} is 0, the same theorem for that reduces to $d\omega^1_2 + \omega^1_3 \wedge \omega^3_2$ and we get the following result.

Theorem 5 (Gauss Curvature Equation)**.** In the above set-up, we have $\Omega^1_2 = \omega^1_3 \wedge \omega^2_3$.

Lemma 5. For any Riemannian 2-fold M, at a fixed $x \in M$, the value $\Omega^1_2(e_1, e_2) \in \mathbb{R}$ for $e_1, e_2 \in T_xM$ an orthonormal basis is independent of the choice of orthonormal basis. $\{e_1, e_2\}$.

Proof. Write $\bar{e} = ea$, where $a = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \in O(2)$. Then $\overline{\Omega}^1_2 = a^{-1}\Omega a = (a^{-1}Ja)\Omega^1_2$, where $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. It is easy to check that if $a \in O(2)$, then $a^{-1}Ja = (\det a)J$, and the result follows from

$$\overline{\Omega}^1_2(\bar{e}_1, \bar{e}_2) = (\det a)\Omega^1_2(\alpha e_1 + \gamma e_2, \beta e_1 + \delta e_2) = (\det a)^2 \Omega^1_2(e_1, e_2) = \Omega^1_2(e_1, e_2).$$

■

Definition 8. This quantity $K_x := \Omega^1_2(e_1, e_2)$ is called the *Gaussian curvature of M at x.*

That this quantity coincides with the one defined using the product of principal curvatures is one of the ways of stating the Theorema Egregium because our definition K_x only depends on the metric. A proof of these facts can be found in [1] §12 as well.

3.3 Examples of Computations: Sphere, Torus

In this section, we end with two explicit computations.

Example 12. Consider $\mathbb{S}^2 \subset \mathbb{R}^3$ with polar coordinates (θ, φ) related to Euclidean coordinates by $x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$ and $z = \cos\theta$. Then an orthonormal frame in these coordinates is given by $e_1 = \partial_\theta, e_2 = (\sin\theta)^{-1}\partial_\varphi$ and $e_3 = e_1 \times e_2$. Then it is easy to see after calculating these out that this orthonormal frame is related to the standard one by $\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix} = \begin{bmatrix} \partial_x & \partial_y & \partial_z \end{bmatrix} a$ where

$$a = \frac{1}{\sqrt{x^2+y^2}} \begin{bmatrix} xz & -y & x\sqrt{x^2+y^2} \\ yz & x & y\sqrt{x^2+y^2} \\ z^2-1 & 0 & z\sqrt{x^2+y^2} \end{bmatrix} \in O(3).$$

By the transformation equation $\overline{\omega} = a^{-1}\omega a + a^{-1}da$ and using the fact that with respect to the standard fram $\omega = 0$, we conclude that the connection matrix here is $\omega = a^{-1}da$. Since $a \in O(3)$, we see that $a^{-1} = a^t$. Therefore after a brief calculation we see that

$$\omega^1_2 = (a^{-1}da)^1_2 = a^1_1 da^1_2 + a^2_1 da^2_2 + a^3_1 da^3_2 = \frac{z}{x^2+y^2}(y\,dx - x\,dy) = -\cos\theta\, d\varphi.$$

Therefore, we get that $\Omega^1_2 = \mathrm{d}\omega^1_2 = \sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi$. In particular, the Gaussian curvature at a point (θ,φ) is simply

$$\Omega^1_2\left(\partial_\theta, \frac{1}{\sin\theta}\partial_\varphi\right) = (\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi)\left(\partial_\theta, \frac{1}{\sin\theta}\partial_\varphi\right) = 1,$$

which says that the round sphere has everywhere constant positive Gaussian curvature of 1.[2]

Example 13. Consider the torus of revolution $\mathbb{T}^2 \subset \mathbb{R}^3$ defined by $(\sqrt{x^2+y^2}-2)^2+z^2 = 1$ with coordinates (θ,φ) related to Euclidean coordinates by $x = (2+\sin\varphi)\cos\theta, y = (2+\sin\varphi)\sin\theta, z = \cos\varphi$. An orthonormal fram for this is given in these coordinates by

$$\begin{aligned}
\mathrm{e}_1 &= \frac{1}{2+\sin\varphi}\frac{\partial}{\partial\theta} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y},\\
\mathrm{e}_2 &= \frac{\partial}{\partial\varphi} = \cos\theta\cos\varphi\frac{\partial}{\partial x} + \sin\theta\cos\varphi\frac{\partial}{\partial y} - \sin\varphi\frac{\partial}{\partial z}, \text{ and}\\
\mathrm{e}_3 &= \sin\varphi\cos\theta\frac{\partial}{\partial x} + \sin\varphi\sin\theta\frac{\partial}{\partial y} + \cos\varphi\frac{\partial}{\partial z}.
\end{aligned}$$

As before, we get a matrix a which in this case is simply

$$a = \begin{bmatrix} -\sin\theta & \cos\theta\cos\varphi & \sin\varphi\cos\theta \\ \cos\theta & \sin\theta\cos\varphi & \sin\varphi\sin\theta \\ 0 & -\sin\varphi & \cos\varphi \end{bmatrix}.$$

Therefore, as before

$$\omega^1_2 = (a^{-1}\mathrm{d}a)^1_2 = a^1_1\mathrm{d}a^1_2 + a^2_1\mathrm{d}a^2_2 + a^3_a\mathrm{d}a^3_3 = \cos\varphi\,\mathrm{d}\theta.$$

Therefore,

$$\Omega^1_2 = \mathrm{d}\omega^1_2 = \sin\varphi\,\mathrm{d}\theta\wedge\mathrm{d}\varphi.$$

In particular, the Gaussian curvature at a point (θ,φ) is simply

$$\Omega^1_2\left(\frac{1}{2+\sin\varphi}\partial_\theta, \partial_\varphi\right) = \frac{\sin\varphi}{2+\sin\varphi}.$$

Therefore, the Gaussian curvature is positive for $\theta \in (0,\pi)$, negative for $\theta \in (\pi, 2\pi)$, and zero for $\theta \in \{0,\pi\}$, all of which make perfect sense geometrically.

[2] Technically, this only works on the open subset on which the parametrization (θ,φ) is valid, but one can use an SO(3)-symmetry argument to fill in the gap.

References

[1] L. W. Tu, *Differential Geometry: Connections, Curvature, and Characteristic Classes*. No. 275 in Graduate Texts in Mathematics, Springer, 2010.

[2] R. W. R. Darling, *Differential Forms and Connections*. Cambridge University Press, 1994.

[3] C. H. Taubes, *Differential Geometry: Bundles, Connections, Metrics and Curvature*. Oxford Graduate Texts in Mathematics, Oxford University Press, 2011.

[4] J. M. Lee, *Introduction to Riemannian Manifolds*. No. 176 in Graduate Texts in Mathematics, Springer, 2018.