Bertrand's Postulate

Why All of Maths is Cool™

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Dramatis Personae

Theorem

For each $n \in \mathbb{Z}_{\geq 2}$*, there is a prime* p *with* $n < p < 2n$.

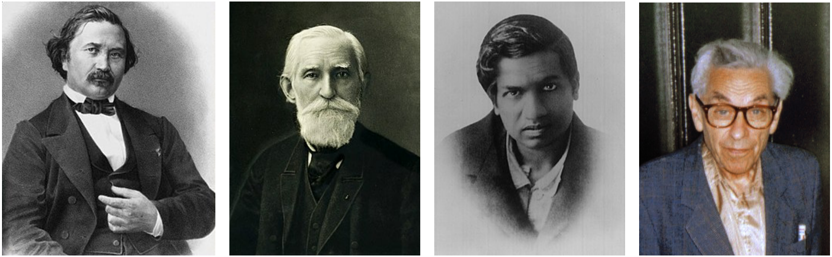

Figure: Some chill guys. Src: Wikipedia.

Bertrand's Postulate/Chebyshev's Theorem

Theorem (Chebyshev)

For each $n \in \mathbb{Z}_{\geq 2}$, there is a prime p with $n < p < 2n$.

Proof idea:

(1) Consider $N = \binom{2n}{n}$. If $\nexists\, p$, then $p \mid N \Rightarrow p \leq n$.

(2) Study asymptotic growth as $x \to \infty$ of the *Chebyshev ϑ-function* $\vartheta : \mathbb{R}_{\geq 1} \to \mathbb{R}_{\geq 0}$ given by

$$\vartheta(x) = \sum_{p \leq x} \log p.$$

(3) Combine (1) and (2) to bound N and hence n from above; say $n \leq B$ for some $B \in \mathbb{R}_{\geq 1}$.

(4) Check up to B "by hand".

Chebyshev ϑ-function

Lemma (1)

For $x \in \mathbb{R}_{\geq 1}$*, let* $\vartheta(x) := \sum_{p \leq x} \log p$*. Then for all* $x \in \mathbb{R}_{\geq 1}$*,*

$$\vartheta(x) < Cx,$$

where $C := \log 4 \sim 1.386\ldots$.

Proof.

(a) For $m \in \mathbb{Z}_{\geq 1}$, have $\vartheta(2m+1) - \vartheta(m+1) < Cm$. Indeed, if $M = \binom{2m+1}{m}$, then $2M < (1+1)^{2m+1}$. If $m+1 < p \leq 2m+1$, then $p \mid M$, and so

$$\vartheta(2m+1) - \vartheta(m+1) = \sum_{m+1 < p \leq 2m+1} \log p \leq \log M < Cm.$$

(b) When $x = n \in \mathbb{Z}_{\geq 1}$ by induction. If $n = 1, 2$, clear. If $n \geq 3$ and $2 \mid n$, then

$$\vartheta(n) = \vartheta(n-1) < C(n-1) < Cn.$$

(c) If $n = 2m+1$ for $m \in \mathbb{Z}_{\geq 1}$, then

$$\vartheta(n) = (\vartheta(2m+1) - \vartheta(m+1)) + \vartheta(m+1) < Cm + C(m+1) = C(2m+1).$$

(d) If $x \in \mathbb{R}_{\geq 1}$, then $\vartheta(x) = \vartheta(\lfloor x \rfloor) < C \lfloor x \rfloor \leq Cx$. □

Bounding the Central Binomial Coefficient

Lemma (2)

If $n \in \mathbb{Z}_{\geq 1}$ and $N = \binom{2n}{n}$, then

(a) We have

$$\log N \geq Cn - \log(2n).$$

(b) For any prime p,

$$v_p(N) \leq \log_p(2n) = \frac{\log 2n}{\log p}.$$

Proof.

(a) N is the largest term in the expansion of $(1+1)^{2n}$, so

$$2^{2n} \leq 2 + \binom{2n}{1} + \cdots + \binom{2n}{n-1} \leq 2nN.$$

(b) By Legendre's formula $v_p(n!) = \sum_{k=1}^{\infty} \lfloor n/p^k \rfloor$, we have

$$v_p(N) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \cdot \left\lfloor \frac{n}{p^k} \right\rfloor \right).$$

Each summand is at most 1, and the summand is zero for $k > \log_p(2n)$. □

Main Proof

Suppose $n \geq 5$, and $\nexists p$. If $N = \binom{2n}{n}$, then for any prime p we have $p \mid N \Rightarrow p \leq n$.

(a) In fact, $p \mid N \Rightarrow p \leq (2/3)n$. Indeed, if $(2/3)n < p \leq n$, then

$$1 \leq \frac{n}{p} < \frac{3}{2} \text{ and } 2 \leq \frac{2n}{p} < 3 \text{ and } p^2 > \frac{4}{9}n^2 > 2n,$$

so

$$v_p(N) = \left\lfloor \frac{2n}{p} \right\rfloor - 2\left\lfloor \frac{n}{p} \right\rfloor = 2 - 2 = 0.$$

(b) If $p^2 \mid N$, then $2\log p \leq v_p(N)\log p \leq \log 2n$ by (2b), and so $p \leq \sqrt{2n}$. Therefore,

$$\sum_{p^2 \mid N} v_p(N)\log p \leq \sqrt{2n}\log 2n.$$

(c) We have $\log N = \sum_{p \| n} \log p + \sum_{p^2 \mid N} v_p(N)\log p \leq \vartheta((2/3)n) + \sqrt{2n}\log 2n$. By (1) and (2a), we have

$$Cn - \log 2n \leq \log N < \frac{2}{3}Cn + \sqrt{2n}\log 2n,$$

so $n \leq B := 467$. But here, $2, 3, 5, 7, 13, 23, 43, 83, 163, 317$, and 631 work. □

Outlook

Applications:

(1) Primes form a complete sequence: any positive integer is a sum of distinct primes (and possibly 1).
(2) For $n \in \mathbb{Z}_{\geq 2}$, the *harmonic number* $H_n = \sum_{k=1}^{n} k^{-1}$ is not an integer.

Addenda:

(1) The prime number theorem ($\pi(x) = \sum_{p \leq x} 1 \sim x(\log x)^{-1}$) implies

$$\forall \varepsilon > 0, \exists\, N \in \mathbb{Z}_{\geq 1} : \forall n \geq N, \exists\, p : n < p < (1+\varepsilon)n.$$

(2) It also implies $\vartheta(x) \sim x$. If RH, then error is $\mathcal{O}(x^{1/2+\varepsilon})$ for any $\varepsilon > 0$.

Takeaways:

(1) Bertrand's Postulate/Chebyshev's Theorem (c.f. Part III *Analytic Number Theory*)
(2) The Millenium Problems (or open problems in general) aren't *too* far away.
(3) You can't separate mathematics (or science) from history.
(4) This place (Trinity College, Cambridge) is *awesome*.
(5) All of Maths is Cool™