Archimedean Local Fields

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Abstract

The purpose of this note is to record four proofs of the result that the only archimedean local fields are $\mathbb{R}$ and $\mathbb{C}$, or slightly more generally that the only complete archimedean valued fields are $\mathbb{R}$ or $\mathbb{C}$ (Gelfand-Tornheim-Ostrowski).

Contents

1 Introduction

In this section we review the definitions of absolute values and prove Ostrowski's Theorem for $\mathbb{Q}$. In the next section, we review the basics of normed vector spaces over valued fields. In the last two sections we give a proof of the claimed result.

Definition 1.1.

(a) An *absolute value* on a field K is a function

$$|\cdot| : K \to \mathbb{R}_{\geq 0}$$

such that
  (i) for all $x \in K$, we have $|x| = 0$ iff $x = 0$,
  (ii) for all $x, y \in K$, we have $|xy| = |x| \cdot |y|$, and
  (iii) for all $x, y \in K$, we have $|x + y| \leq |x| + |y|$.

If the absolute value $|\cdot|$ on K further satisfies that
  (iii') for all $x, y \in K$, we have $|x + y| \leq \max\{|x|, |y|\}$,

then it is said to be *non-archimedean*; else, it is said to be *archimedean*.

(b) For a field K, then *trivial* absolute value is the one given by $|x| = 1$ for all $x \in K^\times$. An absolute value other than this one is said to be *nontrivial*.

(c) A *valued field* is a pair $(K, |\cdot|)$, where K is a field and $|\cdot|$ an absolute value on K; it is said to be (non)-archimedean (resp. (non)trivial) according to whether $|\cdot|$ is.

We will henceforth drop the pedanticity and adopt usual conventions of implicit notation. An absolute value on a field gives it the structure of a topological field. Clearly, the restriction of an absolute value to a subfield defines an absolute value on it. If $(K, |\cdot|)$ is a valued field, then for any root ζ of unity in K we have $|\zeta| = 1$, and so $|-x| = |x|$ for all $x \in K$.

Example 1.2.

(a) On $K = \mathbb{C}$ (and hence on any subfield), the function $|\cdot|_\infty$ given by $a + \mathrm{i}b \mapsto \sqrt{a^2 + b^2}$ for $a, b \in \mathbb{R}$ defines an archimedean absolute value, calle dthe *standard absolute value*.

(b) When $K = \mathbb{Q}$ and p is a prime, then the function $|\cdot|_p$ given by $x \mapsto p^{-v_p(x)}$ when $x \in \mathbb{Q}^\times$ is an absolute value called the *p-adic absolute value* on $\mathbb{Q}$.[1]

Lemma/Definition 1.3. Let K be a field and $|\cdot|$ and $|\cdot|'$ two absolute values on K. The following are equivalent:

(a) There is a $c \in \mathbb{R}_{>0}$ such that $|\cdot|' = |\cdot|^c$.
(b) The induced topologies on K coming from $|\cdot|$ and $|\cdot|'$ are the same.
(c) For all $x \in K$, we have $|x| < 1$ iff $|x|' < 1$.

When these conditions are satisfied, we say that $|\cdot|$ and $|\cdot|'$ are *equivalent*. We say that two valued fields $(K, |\cdot|_K)$ and $(L, |\cdot|_L)$ are *isomorphic* if there is a field isomorphism $\sigma : K \to L$ such that the absolute value $\sigma^*|\cdot|_L$ given on K by $x \mapsto |\sigma x|_L$ for $x \in K$ is equivalen to $|\cdot|_K$.

Proof.

(a) $\Rightarrow$ (b) Clear.
(b) $\Rightarrow$ (c) If $(K, |\cdot|)$ is a valued field, then for $x \in K$ we have $|x| < 1$ iff $\lim_{n\to\infty} x^n = 0$.
(c) $\Rightarrow$ (a) If either of $|\cdot|$ and $|\cdot|'$ is trivial, then so is the other and $c = 1$ works. Else there is an $z \in K$ such that $|z| > 1$, and we need to show that for all $y \in K^\times$ we have

$$\frac{\log |y|}{\log |z|} = \frac{\log |y|'}{\log |z|'}.$$

[1] Here $v_p(x)$ denotes the highest power of p dividing x.

If the left side were less than the right, then there would be $m, n \in \mathbb{Z}$ with $n > 0$ such that m/n squeezes in between them; but then $x := y^n z^{-m}$ has the property that $|x| > 1$ but $|x|' < 1$. This is sufficient by symmetry. ■

Lemma 1.4. A valued field $(K, |\cdot|)$ is nonarchimedean iff $|\cdot|$ is bounded on the image of $\mathbb{Z}$ in K. In particular, if K has positive characteristic, then any absolute value on it is non-archimedean.

Proof. If K is nonarchimedean, then in fact $|n| \leq 1$ for all n in the image of $\mathbb{Z}$ in K. Conversely, suppose that the image of $\mathbb{Z}$ is bounded in absolute value, say by $B \in \mathbb{R}_{>0}$. We need to show that if $x, y \in K^\times$ with $x + y \neq 0$ and $|x| \leq |y|$, then $|x + y| \leq |y|$. For all $n \in \mathbb{Z}_{\geq 1}$,

$$|x + y|^n \leq \sum_{i=0}^{n} \left| \binom{n}{i} \right| |x|^i |y|^{n-i} \leq B(n+1)|y|^n.$$

Extracting n^{th} roots and taking the limit as $n \to \infty$ yields the result. ■

Theorem 1.5 (Ostrowski)**.** Any nontrivial absolute value $|\cdot|$ on $\mathbb{Q}$ is equivalent to exactly one $|\cdot|_p$, where p is either a prime or ∞.

Proof. Suppose first that $|\cdot|$ is non-archimedean, so that by 1.4, we have that $|n| \leq 1$ for all $n \in \mathbb{Z}$. The ideal $\{n \in \mathbb{Z} : |n| < 1\} \subset \mathbb{Z}$ is evidently prime, and it is nonzero because $|\cdot|$ is nontrivial. Therefore, there is a unique prime p such that for $n \in \mathbb{Z}$ we have $|n| < 1$ iff $p \mid n$, and this easily implies (using, e.g., 1.3(c)) that $|\cdot|$ is equivalent to $|\cdot|_p$.

It only remains to analyze the archimedean case in which there is $b \in \mathbb{Z}_{\geq 2}$ such that $|b| > 1$. Let b be the smallest such integer, and let $c \in \mathbb{R}_{>0}$ be such that $|b| = b^c$. It suffices to show that for all $n \in \mathbb{Z}_{\geq 1}$, we have $|n| = n^c$. Let $n \in \mathbb{Z}_{\geq 1}$ be given.

(a) We show that $|n| \leq n^c$. Write the base-b expansion of n as $n = a_0 + a_1 b + \cdots + a_k b^k$ with $k \in \mathbb{Z}_{\geq 0}$ and $a_0, \ldots, a_k \in \{0, 1, \ldots, b-1\}$ such that $a_k \neq 0$. Then $k \leq \log_b n$, and so

$$|n| \leq \sum_{i=0}^{k} |a_i| \cdot |b|^i \leq \sum_{i=0}^{k} b^{ic} = b^{kc} \sum_{i=0}^{k} b^{-ic} \leq b^{kc} \sum_{i=0}^{\infty} b^{-ic} = C b^{kc} \leq C n^c,$$

where $C := (1 - b^{-c})^{-1} \in \mathbb{R}_{>0}$ is independent of n. This holds for each n; replacing n by n^r for $r \in \mathbb{Z}_{\geq 1}$ and extracting r^{th} roots gives us also that $|n| \leq C^{1/r} n^c$. Taking the limit as $r \to \infty$ gives the result.

(b) We show that $|n| \geq n^c$. Let $k := \lfloor \log_b n \rfloor \in \mathbb{Z}$ so that $b^k \leq n < b^{k+1}$. Then

$$|n| \geq |b^{k+1}| - |b^{k+1} - n| \geq b^{(k+1)c} - (b^{k+1} - n)^c \geq b^{(k+1)c} - (b^{k+1} - b^k)^c \geq D n^c,$$

where $D := b^c[1 - (1 - b^{-1})^c] \in \mathbb{R}_{>0}$ is independent of n. The same trick as in (a) finishes the proof. ■

2 Topological Vector Spaces over Valued Fields

Definition 2.1. Let K be a valued field and V a vector space over K. A *norm* on V is a function

$$\|\cdot\| : V \to \mathbb{R}_{\geq 0}$$

such that

(i) for all $v \in V$, we have $\|v\| = 0$ iff $v = 0$,
(ii) for all $x \in K$ and $v \in V$, we have $\|xv\| = |x| \cdot \|v\|$, and
(iii) for all $v, w \in K$, we have $\|v + w\| \leq \|v\| + \|w\|$.

A norm on a vector space V over a valued field K gives it the structure of a topological vector space over K.

Example 2.2.

(a) In the setting of 2.1, if V has finite dimension $n \in \mathbb{Z}_{\geq 1}$ over K and $v_1, \ldots, v_n \in V$ is a K-basis of V, then the function $\|\cdot\|$ on V given by $\|\sum_{i=1}^n x_i v_i\| := \max_{i=1}^n |x_i|$ for $x_1, \ldots, x_n \in K$ is a norm on V called the *sup*-norm.
(b) If L/K is an extension of valued fields (i.e., a field extension in which the absolute value on L extends that on K), then the absolute value on L gives its underlying vector space over K a norm.

Lemma/Definition 2.3. Let K be a valued field, V a vector space over K, and $\|\cdot\|$ and $\|\cdot\|'$ two norms on V. The following are equivalent:

(a) There are $C, D \in \mathbb{R}_{>0}$ such that for all $v \in V$ we have $C\|v\| \leq \|v\|' \leq D\|v\|$.
(b) The induced topologies on V coming from $\|\cdot\|$ and $\|\cdot\|'$ are the same.

Proof. The implication (a) $\Rightarrow$ (b) is clear. For (b) $\Rightarrow$ (a), note that if the induced topologies are same, then there is an $r > 0$ such that for all $v \in V$ we have $\|v\| \leq r$ implies $\|v\|' \leq 1$. Then $D = r^{-1}$ works, and this suffices by symmetry. ■

Theorem 2.4. Let K be a *complete* valued field and V a *finite dimensional* vector space over K. Then any two norms on V are equivalent and V is complete.

Proof. Evidently, V is complete in the sup-norm with respect to any basis (2.2(a)) if K is, so it suffices to prove the first statement. Let $n := \dim_K V$. If $n = 0$, there is nothing to show; hence suppose $n \in \mathbb{Z}_{\geq 1}$, and fix a basis $v_1, \ldots, v_n$ of K. Let $\|\cdot\|$ be any given norm on V. We will show that $\|\cdot\|$ is equivalent to the sup norm with respect to $v_1, \ldots, v_n$, by producing C and D as in 2.3(a).

Let $D := \sum_{i=1}^n \|v_i\|$. Then for any $v \in V$, if we write $v = \sum_{i=1}^n x_i v_i$ with $x_i \in K$, then

$$\|v\| \leq \sum_{i=1}^n |x_i| \|v_i\| \leq D \max_{i=1}^n |x_i|.$$

To produce a C, we proceed by induction on n. When $n = 1$, the constant $C := \|v_1\|$ works. Suppose $n \geq 2$, and for $i = 1, \ldots, n$, let V_i denote the K-span of $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n$. By induction V_i is complete with respect to the restriction of $\|\cdot\|$ to V_i and hence closed in V. Therefore, $S = \bigcup_{i=1}^n (v_i + V_i) \subset V$ is a closed subset not containing 0, so there is a $C \in \mathbb{R}_{>0}$ such that $w \in S$ implies $\|w\| \geq C$. Now given any nonzero $v \in V$, write $v = \sum_{i=1}^n x_i a_i$ and suppose without loss of generality that $|x_1| = \max_{i=1}^n |x_i|$. The result follows from $x_1^{-1} v \in S$.

■

3 The Main Theorem

In this section, we present the main results.

Definition 3.1. A *local field* is a nontrivial valued field that is locally compact.

Lemma 3.2. A local field is complete.

Proof. It suffices to note that all closed balls are compact (since all closed balls are homeomorphic to one another and K is locally compact), and hence sequentially compact (since K is metric). Now a Cauchy sequence must eventually lie in a closed ball. ■

The key result we are interested in is

Theorem 3.3. An archimedean local field is isomorphic as a valued field to either $\mathbb{R}$ or $\mathbb{C}$ with the standard absolute values.

Proof. Let K be an archimedean local field. Since K is archimedean, it has characteristic zero (1.4). By Ostrowski's Theorem (1.5), the restriction of the absolute value of K to its prime subfield $\mathbb{Q}$ is equivalent to the standard absolute value $|\cdot|_\infty$. By 3.2, the completion of $\mathbb{Q}$ in K is isomorphic to $\mathbb{R}$ with its standard absolute value. The absolute value on K makes it a locally compact topological vector space over $\mathbb{R}$ (2.2(b)), whence $[K : \mathbb{R}] < \infty$ ([3, Theorem 1.22]). By the Fundamental Theorem of Algebra, this is possible only if $K = \mathbb{R}$ or $K = \mathbb{C}$. In the former case, there is nothing left to show. In the latter case, the absolute value on K defines a norm on the underlying $\mathbb{R}$-vector space (2.2(b)) of K, which by 2.4 is equivalent as a norm to the one coming from the standard absolute value on $\mathbb{C}$. It follows from 2.3 and 1.3 that it is equivalent as an absolute value to the standard absolute value on $\mathbb{C}$. ■

In fact, the "locally compact" condition is stronger than needed, and we have

Theorem 3.4 (Gelfand-Tornheim-Ostrowski)**.** A complete archimedean valued field is isomorphic as a valued field to either $\mathbb{R}$ or $\mathbb{C}$ with the standard absolute values.

Note that 3.2 and 3.4 imply 3.3. We will now give three proofs of 3.4, the first two of which are taken from [2]. The final proof, taken from [1], is almost provided in this section as well, except for one technical detail which we address in the final section.

Proof 1 of 3.4. ([2, §1.2.9]) Let K be such a field; proceeding as in the proof of 3.3, it remains to show that $K/\mathbb{R}$ is algebraic. For this, for $z \in \mathbb{C}$, let $p_z(t) := t^2 - (z + \overline{z})t + z\overline{z} \in \mathbb{R}[t]$ and for $\alpha \in K$, define the function $f : \mathbb{C} \to \mathbb{R}_{\geq 0}$ by $f(z) := |p_z(\alpha)|$ for $z \in \mathbb{C}$. Then f is a continuous proper map and so attains a minimum value say $m \in \mathbb{R}_{\geq 0}$. We will show that $m = 0$.

Since the level set $Z := f^{-1}(m)$ is compact, there is a $z_0 \in Z$ of maximal $|z_0|_\infty$. If $m > 0$, then pick an $\varepsilon \in \mathbb{R}_{>0}$ such that[2] $|\varepsilon| < m$ and consider a root w of $p_{z_0}(t) + \varepsilon$; then $w \in \mathbb{C} \smallsetminus \mathbb{R}$ and

$$|z_0|_\infty = \sqrt{|w|_\infty^2 - \varepsilon} < |w|_\infty,$$

which implies $f(w) > m$. We will show that also $f(w) \leq m$, which is the required contradiction.

Indeed, for odd $n \in \mathbb{Z}_{\geq 1}$, if we factor the polynomial $q_n(t) := p_{z_0}(t)^n + \varepsilon^n \in \mathbb{R}[t]$ over $\mathbb{C}$ as $q_n(t) = \prod_{i=1}^{2n}(t - w_i)$ with $w_1, \dots, w_{2n} \in \mathbb{C}$, then after renumbering we may assume that

[2] This we may do because $|\cdot|$ restricted to $\mathbb{R}$ is equivalent to $|\cdot|_\infty$. If it makes one more comfortable, they may take $\varepsilon \in \mathbb{Q}_{>0}$.

$w = w_1$. Since $q_n(t) \in \mathbb{R}[t]$, we have that

$$q_n(t)^2 = \prod_{i=1}^{2n} p_{w_i}(t)$$

and hence

$$|q_n(\alpha)|^2 = \prod_{i=1}^{2n} f(w_i) \geq f(w) \cdot m^{2n-1}.$$

But now also

$$|q_n(\alpha)| \leq |f(z_0)|^n + |\varepsilon|^n = m^n + |\varepsilon|^n,$$

and so

$$f(w) \leq \frac{|q_n(\alpha)|^2}{m^{2n-1}} \leq \frac{(m^n + |\varepsilon|^n)^2}{m^{2n-1}} = m\left(1 + \left(\frac{|\varepsilon|}{m}\right)^n\right)^2.$$

Taking the limit as $n \to \infty$ yields the result. ■

Proof 2 of 3.4. ([2, §1.3.3]) Let K be a complete archimedean valued field. As in the proof of 3.3, K contains $\mathbb{R}$ in such a way that the restriction of the absolute value on K to $\mathbb{R}$ is equivalent to the standard absolute value on $\mathbb{R}$. We now have two cases:

(a) If there is a square root of -1 in K, then in fact K contains $\mathbb{C}$, and by the same argument as in the proof of 3.3, the restriction of the absolute value on K to $\mathbb{C}$ is equivalent to the standard absolute value. But then K is a complex Banach algebra which is a field, so that by the Gelfand-Mazur Theorem ([3, Theorem 10.14]) we have $K = \mathbb{C}$.

(b) If there does not exist a square root of -1 in K, then we consider the quadratic field extension $K[i] := K[t]/(t^2 + 1)$, which admits the structure of a $\mathbb{C}$-algebra. In this case, the norm $\|\cdot\|$ on $K[i]$ given by $\|x + \mathrm{i}y\| := |x| + |y|$ may not be an absolute value on $K[i]$, but it certainly does make $K[i]$ a complex Banach algebra, and then the same argument as above works.

■

Proof 3 of 3.4. ([1, Chapter 3]) We proceed as in Proof 2, but handle the cases differently.

(a) We proceed very similarly to Proof 1, but taking $p_z(t) := t - z$ this time, so $f(z) = |\alpha - z|$. Let m, Z, z_0 be as before, and pick $\varepsilon \in \mathbb{R}_{>0}$ so that $|\varepsilon| < m$. Let $w := z_0 \pm \varepsilon$, with the sign chosen so that $|w|_\infty > |z_0|_\infty$, which implies $f(w) > m$. This time, for odd $n \in \mathbb{Z}_{\geq 1}$, we set $q_n(t) := p_{z_0}(t)^n \mp \varepsilon^n$ and proceed similarly to that proof to obtain $f(w) \leq m(1 + |\varepsilon|^n m^{-n})$. Taking the limit as $n \to \infty$ finishes the proof.

(b) This time, we show that $K[i]$ can be made into a local field by introducing an absolute value whose restriction to K is equivalent to the given absolute value. Since this is somewhat technical and best developed in a slightly different framework, we postpone this to the next section. Given this, we are done since $K[i]$ is archimedean (by, say, 1.4) and so by (a) we have $K[i] = \mathbb{C}$, proving $K = \mathbb{R}$.

■

4 Generalized Absolute Values and Artin Constants

In this section, we discuss *generalized* absolute values (which [1] calls *valuations*), and see how they give a slightly cleaner approach to dealing with absolute values. In particular, we use this to give a proof of the assertion that if K is a complete valued field which does not have a square root of -1, then there is an absolute value on $K[i]$ whose restriction to $K$is equivalent to the given absolute value.

Definition 4.1. A *generalized absolute value* on a field K is a function

$$|\cdot| : K \to \mathbb{R}_{\geq 0}$$

that satisfies properties (i) and (ii) of 1.1 and also that

(iii") there is a $C \in \mathbb{R}_{>0}$ such that for all $x \in K$, we have $|x| \leq 1$ implies $|1+x| \leq C$.

The infimum of all possible C in (iii") is called the *Artin constant* of $|\cdot|$.

Equivalently, the Artin constant of $|\cdot|$ is the smallest $C \in \mathbb{R}_{>0}$ such that for all $x, y \in K$ we have $|x+y| \leq C \max\{|x|, |y|\}$ (c.f. 4.2(a)). It is clear that if $|\cdot|$ is a generalized absolute value on a field K, then so is $|\cdot|^c$ for each $c \in \mathbb{R}_{>0}$, something which is *not* true of absolute values in general.

Lemma 4.2. Let $|\cdot|$ be a generalized absolute value on a field K with Artin constant C.

(a) We have $C \geq 1$.
(b) The function $|\cdot|$ defines an absolute value iff $C \leq 2$.
(c) Further, in (b), the resulting absolute value is nonarchimedean iff $C = 1$.

Proof.

(a) By (i) and (ii), we have $|0| = 0$ and $|1| = 1$. The result follows by taking $x = 0$ in (iii").
(b) Clearly, (iii) implies (iii") with $C = 2$. Conversely, if for all $x, y \in K$ we have that $|x+y| \leq 2\max\{|x|, |y|\}$, then by induction, for each $N \in \mathbb{Z}_{\geq 1}$ of the form 2^n for some $n \in \mathbb{Z}_{\geq 0}$ and all $x_1, \ldots, x_N \in K$, we have

$$|x_1 + \cdots + x_N| \leq N \max_{i=1}^{N} |x_i|.$$

When N is not a power of 2, let $n := \lceil \log_2 N \rceil$ so that $2^{n-1} < N \leq 2^n$, and fill the empty spots with zeroes (i.e., set $x_{N+1} = \cdots = x_{2^n} = 0$) to obtain the weaker

$$|x_1 + \cdots + x_N| \leq 2N \max_{i=1}^{N} |x_i|.$$

In particular, for any $N \in \mathbb{Z}_{\geq 1}$, taking $x_1 = \cdots = x_N$ gives us $|N| \leq 2N$. Then for any $x, y \in K$ and $n \in \mathbb{Z}_{\geq 1}$,

$$|x+y|^n \leq 2(n+1) \max_{i=0}^{n} \left| \binom{n}{i} x^i y^{n-i} \right| \leq 4(n+1) \max_{i=0}^{n} \binom{n}{i} |x|^i |y|^{n-i} \leq 4(n+1)(|x|+|y|)^n.$$

Extracting n^{th} roots and taking the limit as $n \to \infty$ yields the result.
(c) Clear, since (iii') is directly equivalent to (iii") for $C = 1$.

■

Remark 1. In fact, one can show along very similar lines to 4.2(b) that that the Artin constant C is actually just $C = \max\{1, |2|\}$, so that the function $|\cdot|$ defines an absolute value iff $|2| \leq 2$ and a nonarchimedean one iff $|2| \leq 1$. See [2].

Now we can finish Proof 3 of 3.4.

Lemma 4.3. Let K be a complete valued field and suppose that there is no square root of -1 in K.

(a) There is a $\Delta \in \mathbb{R}_{>0}$ such that for all $x, y \in K$ we have $|x^2 + y^2| \geq \Delta \max\{|x|^2, |y|^2\}$.
(b) There is an absolute value on $K[i]$ whose restriction to K is equivalent to the given absolute value.

Proof.

(a) In fact, $\Delta := |4|(1 + |4|)^{-1}$ will do. Suppose there is an $a_0 \in K$ such that $|a_0^2 + 1| < \Delta$, and set

$$\varepsilon := \frac{|a_0^2 + 1|}{|4|(1 - |a_0^2 + 1|)},$$

so that $0 < \varepsilon < 1$. It is then easy to check that the sequence a_n in K defined by

$$a_{n+1} = a_n - \frac{a_n^2 + 1}{2a_n}$$

for $n \geq 1$ is well-defined, satisfies that for all $n \geq 0$

$$|a_n^2 + 1| \leq \varepsilon^{2^n - 1} |a_0^2 + 1|,$$

and is a Cauchy sequence. Its limit a then satisfies $a^2 + 1 = 0$, i.e., is a square root of -1, a contradiction to hypothesis.

(b) We will show that the function $|\cdot|$ on $K[i]$ given by $x + \mathrm{i}y \mapsto |x^2 + y^2|^{1/2}$ for $x, y \in K$ is a generalized absolute value on K. It follows that for some $c \in \mathbb{R}_{>0}$, the generalized absolute value $x + \mathrm{i}y \mapsto |x^2 + y^2|^{c/2}$ has Artin constant $C \leq 2$, and so by 4.2(b) defines an absolute value, the restriction of which to K is evidently equivalent to the given one. Indeed, property (i) follows from the fact that there is no square root of -1 in K and property (ii) uses the Euler identity

$$(xz - yw)^2 + (yz + xw)^2 = (x^2 + y^2)(z^2 + w^2) \in \mathbb{Z}[x, y, z, w].$$

Itremains to show (iii"), so suppose we are given $x, y \in K$ such that $|x^2 + y^2| \leq 1$. If Δ is as in (a), then

$$\max\{|x|, |y|\} \leq \Delta^{-1/2}.$$

Then

$$|1 + x + \mathrm{i}y|^2 = |(1 + x)^2 + y^2| \leq 1 + |2| \cdot |x| + |x|^2 + |y|^2 \leq 1 + |2|\Delta^{-1/2} + 2\Delta^{-1}$$

and so (iii") is satisfied for

$$C := (1 + |2|\Delta^{-1/2} + 2\Delta^{-1})^{1/2}.$$

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