2.5 Exercise Sheet 5

2.5.1 Standard Exercises/Numerical and Exploration

Exercise 2.5.1. Given a nonempty finite set $S \subset \mathbb{P}^2_k$ of points in $\mathbb{P}^2_k$, let $d(S)$ be the smallest degree of a curve $C \subset \mathbb{P}^2_k$ through S, i.e. such that $C \supset S$. Let's investigate the relationship between S, its size $n := \#S$, and the integer $d(S)$.

(a) Show that if $n \in \{1, 2\}$, then $d(S) = 1$.
(b) Show that if $n \in \{3, 4\}$, then $d(S) \in \{1, 2\}$. When does each case hold?
(c) Show that if $n = 5$, then $d(S) \in \{1, 2\}$, or equivalently that given any five distinct points $P_1, \ldots, P_5 \in \mathbb{P}^2_k$, there is at least one (possibly reducible) conic $C \subset \mathbb{P}^2_k$ passing through each P_i.
(d) Show that, in general, we have

$$1 \leq d(S) \leq \left\lceil \frac{\sqrt{9 + 8n} - 3}{2} \right\rceil,$$

where $\lceil \cdot \rceil$ denotes the ceiling function. (Hint: When does a system of N linear equations in M variables always have a solution that is not identically zero?)
(e) (Cramer's Theorem) Show that the bound in (d) is sharp in general: for each $n \geq 1$, come up with a collection S of n points such that $d(S)$ equals the upper bound from (d). Can you characterize the sets S for which this equality holds? What possible intermediate values of $d(S)$ are possible?

Exercise 2.5.2. Let k be a field.

(a) Suppose $\operatorname{ch} k \neq 2$, and consider the collection of 9 points $S := \{(i, j) \in \mathbb{A}^2_k : 0 \leq i, j \leq 2\}$. How many distinct cubic curves $C \subset \mathbb{A}^2_k$ pass through S? (Hint: by Exercise 2.5.1(d), there is at least one such C. Does your answer change if the question is about projective cubics instead? Does the choice of base field matter? Can you come up with an analog if $\operatorname{ch} k = 2$?)
(b) Can you formulate an analog of (a) for a configuration of n^2 points

$$S := \{(i, j) \in \mathbb{A}^2_k : 0 \leq i, j \leq n - 1\},$$

where $n \geq 2$ is any integer (say when $\operatorname{ch} k = 0$ for convenience)?

Exercise 2.5.3 (More on Pascal). (Adapted from [3, Exercise 5.31].) If in Pascal's Theorem, we let some adjacent vertices coincide (the side being tangent), then we get many new theorems.

(a) State and sketch what happens if $P_1 = P_2$, $P_3 = P_4$ and $P_5 = P_6$.
(b) Let $P_1 = P_2$ and the other four points be distinct. Deduce a rule for constructing a tangent to a given conic at a given point, using only a straight-edge.

Exercise 2.5.4. Let $C \subset \mathbb{P}^2_k$ be a curve of degree d over an algebraically closed field k.

(a) Make sense of the following statement: a "general" line $L \subset \mathbb{P}^2_k$ intersects C in exactly d distinct points.
(b) Given a "general" point $P \in \mathbb{P}^2_k$, how many lies through P are tangent to C?

(Hint: How is this exercise is related to Exercises 2.5.8, 2.5.9, and 2.5.10? For (b), you may suppose for convenience that $\operatorname{ch} k = 0$. What happens in positive characteristic?)

Exercise 2.5.5. Let k be an algebraically closed field, and let $C \subset \mathbb{P}^2_k$ be a smooth cubic curve.

(a) Show that C has exactly 9 inflection points. The set of inflection points on C is usually denoted by $C[3]$. (Hint: Exercise 2.4.5. You may assume $\operatorname{ch} k \neq 2, 3$ for convenience.)

(b) Show that $C[3]$ is not contained in a line, but any line passing through any two points in $C[3]$ passes through a third point in $C[3]$. Why does this not violate the Sylvester-Gallai Theorem?

(c) Suppose that $\operatorname{ch} k \neq 3$. Show that by a projective change of coordinates, we can bring $C[3]$ to be the nine points

$$[0:1:\xi], [\xi:0:1], [1:\xi:0],$$

where ξ runs over the three roots of $t^3+1=0$ in k.[9]

(d) Keeping the hypothesis that $\operatorname{ch} k \neq 3$, show that every cubic curve passing through the 9 points from (c) has the equation

$$F_\Lambda = \lambda(X^3+Y^3+Z^3) + 3\mu XYZ \in k[X,Y,Z]$$

for some $\Lambda := [\lambda : \mu] \in \mathbb{P}^1_k$. This curve is singular iff Λ is either $[0:1]$ or $[1:\xi]$ where $\xi^3+1=0$. In each of these cases the curve $C_\Lambda := C_{F_\Lambda}$ degenerates into a product of three lines. If C_Λ is irreducible, then the flexes of C_Λ are exactly the 9 points above.

(e) Conclude, using either (b) or both (c) and (d), that if $k = \mathbb{C}$, then

$$\#\left(C[3] \cap C(\mathbb{R})\right) \leq 3,$$

i.e. at most three of the flexes of a complex smooth cubic curve can be real. Come up with a curve C for which this bound is achieved. Can this intersection have fewer than 3 points? Can it have exactly 2?

Exercise 2.5.6. If $f, g \in k[x,y]$ are nonconstant polynomials and $P \in \mathbb{A}^2_k$, then

$$i_P(f,g) \geq m_P(f) \cdot m_P(g).$$

When does equality hold? (This is a very hard exercise, and you may not be able to do it with the tools we have developed so far; nonetheless, it is very valuable to work out special cases. Try doing the case when f or g is linear. Next, try the case when $m_P(f) = 1$ or $m_P(g) = 1$. Finally, see how far you can extend your techniques to the next (or general) case; once you've done that, see [3, §3.3, Theorem 3] or [15, Theorem 7.4].)

2.5.2 PODASIPs

Prove or disprove and salvage if possible the following statements.

Exercise 2.5.7 (Braikenridge-Maclaurin Theorem/Converse to Pascal's Theorem)**.** If the intersection points of opposite sides of a hexagon lie on a straight line, then the vertices of the hexagon lie on a conic.

Exercise 2.5.8. (Adapted from [3, Exercise 5.26].) If $C \subset \mathbb{P}^2_k$ is a curve of degree $n \geq 1$, and $P \in \mathbb{P}^2_k$ a point of multiplicity $m := m_P(C) \geq 0$, then for all but finitely many lines L through P, the line L intersects C in $n - m$ distinct points other than P.

Exercise 2.5.9. Given a curve $C \subset \mathbb{P}^2_k$ and a point $P \in \mathbb{P}^2_k$, there is at least one tangent line L to C that does not pass through P.

Exercise 2.5.10 (Dual Curve)**.** Let $C \subset \mathbb{P}^2_k$ be a curve. Let

$$C^* := \{L \in \mathbb{P}^{2*}_k : L \text{ is tangent to } C \text{ at some point } P \in C\} \subset \mathbb{P}^{2*}_k.$$

Then $C^* \subset \mathbb{P}^{2*}_k$ is a curve, and $C^{**} = C$. (Hint: Can you work out a few examples in low degrees? What is the relationship between the degrees of C and C^*?)

[9]That these roots are distinct uses $\operatorname{ch} k \neq 3$.