

1.8 06/26/24 - Smoothness, Multiplicity, Tangent Lines

Today, we will talk about smoothness of algebraic curves. What should smoothness mean—i.e. what should it mean to say that a curve $C \subset \mathbb{A}_k^2$ is smooth at a point $P \in C$? One definition is that at each point, we have a well-defined tangent direction, i.e. that the curve is well-approximated by a linear polynomial. Certainly, whatever this notion is, it should be invariant under affine changes of coordinates, so we may focus on the case when $P = (0, 0)$, and then considering a few examples naturally leads us to the following definition.

Definition 1.8.1.

- (a) A polynomial $f(x, y) \in k[x, y]$ is said to be **homogeneous of degree $d \geq 0$** if in the ring $k[x, y, t]$, we have the polynomial identity

$$f(tx, ty) = t^d f(x, y).$$

This is equivalent to saying that in an expression of the form $f(x, y) = \sum_{i,j \geq 0} a_{i,j} x^i y^j$ with $a_{i,j} \in k$, we have $a_{i,j} = 0$ unless $i + j = d$. For each $d \geq 0$, the set of all polynomials in $k[x, y]$ of degree d will be denoted by $k[x, y]_d$.

- (b) Any $f(x, y) \in k[x, y]$ can be written uniquely as

$$f = f_0 + f_1 + \cdots + f_d,$$

where $d = \deg f \geq 0$, and for each i with $0 \leq i \leq d$, the polynomial $f_i \in k[x, y]$ is homogeneous of degree i . If $0 \neq f$, then there is a unique smallest index i_0 such that $f_{i_0} \neq 0$; in this case, we define the **multiplicity of f at the origin $O = (0, 0)$** , written $m_O(f)$, and the **initial part of f** , written $\text{in}(f)$, to be, respectively,

$$m_O(f) = i_0 \text{ and } \text{in}(f) := f_{i_0}.$$

Example 1.8.2. If $f(x, y) = y^2 - x^3$, then $m_O(f) = 2$ with $\text{in}(f) = y^2$.

We say that a function $F : \mathbb{A}_k^2 \rightarrow k$ is homogeneous of degree $d \geq 0$ if for all $(p, q) \in \mathbb{A}_k^2$ and $t \in k$, we have $F(tp, tq) = t^d F(p, q)$. If a polynomial $f \in k[x, y]$ is homogeneous of degree $d \geq 0$, then so is the associated function F_f , and the converse holds if k is infinite. Note that the zero polynomial $0 \in k[x, y]$ is homogeneous of degree d for every $d \geq 0$, and for each $d \geq 0$, the subset $k[x, y]_d \subset k[x, y]$ is a vector subspace of dimension $d + 1$ with basis $x^d, x^{d-1}y, \dots, xy^{d-1}, y^d$, with $k[x, y] = \bigoplus_{d \geq 0} k[x, y]_d$. Finally, if $f \in k[x, y]_d$ and $g \in k[x, y]_e$, then $fg \in k[x, y]_{d+e}$. This structure on $k[x, y]$ is called the structure of a **graded k -algebra**.

Lemma 1.8.3. If $k = \bar{k}$, then for any $d \geq 0$ and $f \in k[x, y]_d$, there are homogeneous linear polynomials $\ell_1, \dots, \ell_d \in k[x, y]_1$ such that $f = \ell_1 \ell_2 \cdots \ell_d$. If f is nonzero, then these factors are uniquely determined up to reordering and nonzero scalars.

Proof. Write $f = \sum_{i=0}^d a_i x^{d-i} y^i$. If $f \neq 0$, let i_0 be the least index such that $a_{i_0} \neq 0$. Since $k = \bar{k}$, we can factor the polynomial $f(t, 1) = \sum_{i=i_0}^d a_i t^{d-i}$ of degree $d - i_0$ as

$$f(t, 1) = \sum_{i=i_0}^d a_i t^{d-i} = a_{i_0} \prod_{j=1}^{d-i_0} (t - \alpha_j)$$

for some $\alpha_j \in k$, and then taking $a_{i_0}^{-1} \ell_1 = \ell_2 = \cdots = \ell_{i_0} = y$ and $\ell_{i_0+j} = x - \alpha_j y$ for $j = 1, \dots, d - i_0$ suffices. Uniqueness is clear because $k[x, y]$ is a UFD, and each ℓ_j is prime. ■

Definition 1.8.4.

- (a) Given a curve $C \subset \mathbb{A}_k^2$, we define the multiplicity of C at the origin $O = (0, 0)$ to be

$$m_O(C) := m_O(f_C),$$

where $f_C \in k[x, y]$ is any minimal polynomial for C . If $\text{in}(f_C) = \ell_1 \cdots \ell_m$ is the factorization of $\text{in}(f_C)$ into linear factors as in Lemma 1.8.3 where $m := m_O(C)$, then we define the tangent lines to C at O to be the lines $L_j := C_{\ell_j}$ for $j = 1, \dots, m$. (These need not all be distinct, and are independent of the choice of f_C .) Finally, the tangent cone to C at O is define to be

$$\text{TC}_O C := C_{\text{in}(f)} = L_1 \cup L_2 \cup \cdots \cup L_m.$$

- (b) Given a curve $C \subset \mathbb{A}_k^2$ and an arbitrary point $P \in \mathbb{A}_k^2$, we define the multiplicity of C at P to be

$$m_P(C) := m_O(\phi^{-1}C),$$

where $\phi : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$ is any affine change of coordinates such that $\phi(O) = P$. We define the tangent lines to C at P to be the lines $\phi(L_j)$ for $j = 1, \dots, m$ where $m = m_P(C)$, and similarly the tangent cone to C at P to be

$$\text{TC}_P C = \phi(\text{TC}_O(\phi^{-1}C)).$$

- (c) Given a curve $C \subset \mathbb{A}_k^2$ and point $P \in \mathbb{A}_k^2$, we have $m_P(C) \geq 1$ iff $P \in C$, in which case we say that P is a smooth point of C iff $m_P(C) = 1$. The curve C is said to be smooth if every $P \in C$ is a smooth point. A point $P \in C$ that is not a smooth point is called a singular point or multiple point of C .^a

^aOutside of mathematics, the terms “singular” and “multiple” are usually antonyms; in this case, they are not, because “singular” here means “exceptional” or “extraordinary” (see Theorem 1.9.7), while “multiple” means “of higher (i.e. > 1) multiplicity”.

Note that a smooth point on a curve has a unique tangent line, which we will denote by $\text{TP}_P C$. The coordinate-invariance of smoothness and multiplicity is baked into the definition—if we can show that it is well-defined. To do this, we need that if $\phi : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$ is an affine change of coordinates such that $\phi(O) = O$, then for any polynomial $f \in k[x, y]$ we have $m_O(f) = m_O(\phi^*(f))$. By considering the homogeneous parts separately, this reduces to showing

Lemma 1.8.5. If $\phi : \mathbb{A}_k^2 \rightarrow \mathbb{A}_k^2$ is an affine change of coordinates such that $\phi(O) = O$, and if $0 \neq f \in k[x, y]$ is homogeneous of degree $n \geq 0$, then so is $\phi^*(f)$.

Proof. Note that ϕ is of the form $\phi(x', y') = (ax' + by', cx' + dy')$ for some $a, b, c, d \in k$ with $ad - bc \neq 0$. The claim is clear when $n = 0$, since then f is a nonzero constant and $\phi^*(f) = f$. When $n = 1$, we have $f = \lambda x + \mu y$ for some $\lambda, \mu \in k$, not both zero, and then

$$\phi^*(f) = \lambda(ax' + by') + \mu(cx' + dy') = (a\lambda + c\mu)x' + (b\lambda + d\mu)y'.$$

Now, since one of λ and μ is not zero, and since $ad - bc \neq 0$, it follows easily that at least one of $a\lambda + c\mu$ and $b\lambda + d\mu$ is nonzero (this is basic linear algebra, but can also be shown directly—how?). Therefore, we are done in this case. If $n \geq 2$, then by Lemma 1.8.3 we can write $f = \ell_1 \cdots \ell_n$ for some ℓ_j homogeneous of degree 1; then we are done by the case $n = 1$ and the observation $\phi^*(f) = \phi^*(\ell_1)\phi^*(\ell_2) \cdots \phi^*(\ell_n)$. This finishes the proof when $k = \bar{k}$ (which is the only case we care about), but in general, we can use Theorem 1.4.5 to reduce to this case. ■

Example 1.8.6. The parabola C defined by $f(x, y) = y - x^2 \in k[x, y]$ has is smooth at the point $(1, 1) \in \mathbb{A}_k^2$ with tangent line L defined by the vanishing of $y - 2x + 1 = 0$.

Example 1.8.7. A curve C is said to have a **simple node** at P iff $m_P(C) = 2$ and C has two distinct tangent lines at P . For instance, the curve C defined by $f(x, y) = y^2 - x^2(x+1) \in k[x, y]$ over a field k with $\text{ch } k \neq 2$ has a simple node at the origin, with tangent lines L_1, L_2 defined by the vanishing of $y \pm x$, and tangent cone $T_O(C) = L_1 \cup L_2$. (What happens when $\text{ch } k = 2$?)

Of course, this definition is not very convenient when we want to locate all singular points of a given curve C . For this, we need a more convenient criterion. This is provided by

Theorem 1.8.8 (Affine Jacobi Criterion). Suppose we are given a curve $C \subset \mathbb{A}_k^2$ and a point $P = (p, q) \in \mathbb{A}_k^2$. Let $f \in k[x, y]$ be a minimal polynomial for C . Then

- (a) $P \in C$ iff $f|_P := f(p, q) = 0$, and in this case
- (b) P is a singular point of C iff

$$\left. \frac{\partial f}{\partial x} \right|_P = \left. \frac{\partial f}{\partial y} \right|_P = 0.$$

- (c) If $P \in C$ is a smooth point, then the tangent line T_PC is defined by the vanishing of

$$\left. \frac{\partial f}{\partial x} \right|_P (x - p) + \left. \frac{\partial f}{\partial y} \right|_P (y - q) \in k[x, y].$$

Wait, what? What are these partial derivative symbols? Why can we do this over any field k ? We'll discuss this more next time, but for now let's work out an example to see how conveniently Theorem 1.8.8 allows us to locate singular points of a curve C .

Example 1.8.9. If $f(x, y) = y - x^2$, then $\partial f / \partial y \equiv 1$ tells us that f is smooth everywhere. At the point $P = (t, t^2)$, the tangent line to C is given by the vanishing of

$$-2t(x - t) + 1(y - t) = y - 2tx + t^2 \in k[x, y].$$

Note that when $\text{ch } k = 2$, this tangent line is always horizontal—which is incredibly weird. In general, weird stuff happens to curves of degree p in characteristic p —watch out for this over the next few weeks!

Example 1.8.10. If $f(x, y) = y^2 - x^3$, then the system of equations we need to solve for the singular points of C is

$$\begin{aligned} y^2 - x^3 &= 0, \\ -3x^2 &= 0, \\ 2y &= 0, \end{aligned}$$

which in any characteristic has the unique solution $(x, y) = (0, 0)$ (check!). Therefore, the unique singular point of C is the origin O , where C has the unique tangent line $y = 0$, i.e. the x -axis.