1.4 06/17/24 - Changes of Coordinates, Nonempty Curves

1.4.1 Affine Changes of Coordinates

Definition 1.4.1. An affine change of coordinates is a transformation

$$\phi : \mathbb{A}^2_k(x', y') \to \mathbb{A}^2_k(x, y)$$

of the form

$$(x, y) = \phi(x', y') = (ax' + by' + p, cx' + dy' + q),$$

for some $a, b, c, d, p, q \in k$, where $ad - bc \neq 0$.

Here $\mathbb{A}^2_k(x', y')$ is just the plane $\mathbb{A}^2_k$, which we think of as having coordinates x', y' (and similarly for $\mathbb{A}^2_k(x, y)$). The $ad - bc \neq 0$ condition guarantees that ϕ is invertible (why?). Affine changes of coordinates comprise of a linear map following by a translation; in particular, the image $\phi(0, 0) = (p, q)$ of the "origin" $(0, 0) \in \mathbb{A}^2_k$ can be any point, i.e. all points look the same (see also Remark 1.1.18).

Note that such a transformation induces a map on the polynomial rings in the opposite direction, i.e. we have a ring homomorphism (even a k-algebra homomorphism)

$$\phi^* : k[x, y] \to k[x', y'], \quad x \mapsto ax' + by' + c, y \mapsto cx' + dy' + q$$

which records the same information. For instance, ϕ is an isomorphism iff ϕ^* is. The reason for this switching of direction, also called "contravariance," is that you should think of $k[x, y]$ as the ring of polynomial functions $f : \mathbb{A}^2_k \to k$, so a coordinate transformation $\phi : \mathbb{A}^2_k(x', y') \to \mathbb{A}^2_k(x, y)$, or more properly ϕ^*, takes a function $f : \mathbb{A}^2_k(x, y) \to k$ to the function

$$\phi^* f = f \circ \phi : \mathbb{A}^2_k(x', y') \to k$$

obtained via precomposition. (This is the ultimate root of all contravariance in algebraic geometry.) Of course, thinking of polynomials as functions is not *quite* right, as you are invited to explore in Exercise 2.2.6; however, this suffices to get good intuition.

Here are a few things you can do with these: check that given any point $(p, q) \in \mathbb{A}^2_k$ and line ℓ through (p, q), there is an affine change of coordinates $\phi : \mathbb{A}^2_k(x', y') \to \mathbb{A}^2_k(x, y)$ such that $\phi(0, 0) = (p, q)$ and $\phi^{-1}\ell = C_x$, i.e. such that in the coordinate system (x', y'), the point (p, q) moves to the origin and the line ℓ moves to the y-axis C_x. We shall often define things in this course in good coordinate systems–it is then *your* job to check that these definitions are invariant under affine changes of coordinates. You are invited to play with the transformation of conics under affine changes of coordinates in Exercise 2.1.6.

1.4.2 Algebraically Closed Fields

As we have seen many times previously, it may very well happen over an arbitrary (even infinite) field k that the vanishing locus $C_f \subset \mathbb{A}^2_k$ of a polynomial function corresponding to a nonconstant polynomial $f \in k[x, y]$ is just empty. One example of this situation is when

$$f(x, y) = x^n + a_1 x^{n-1} + \cdots + a_n \in k[x, y],$$

i.e. that f is a polynomial of x alone. In this case, the corresponding locus C_f is nonempty iff this equation has a root in k, in which case C_f is the union of some vertical lines (see Remark 1.1.12). This suggests that the problem lies already in finding solutions to polynomial in one variable.

Definition 1.4.2. A field k is said to be algebraically closed if for every nonconstant polynomial $f(x) \in k[x]$, there is a root of f in k, i.e. there is an $\alpha \in k$ such that $f(\alpha) = 0$.

Example 1.4.3. The fields $\mathbb{Q}, \mathbb{R}$ and $\mathbb{F}_q$ for any q are not algebraically closed (why?).

Here are two facts which I will take for granted–these are important theorems in their own right, but this course is perhaps not the right place for them.

Theorem 1.4.4 (Fundamental Theorem of Algebra)**.** The field $\mathbb{C}$ is algebraically closed.

Theorem 1.4.5. Given any field k, there is an algebraically closed field k' containing k.

Theorem 1.4.5 says that every field k can be embedded into some algebraically closed one, although in many different ways in general.[10] This theorem says that we lose little when passing to algebraically closed fields, even when working in positive characteristic. The "smallest"[11] algebraically closed field containing k is often called the algebraic closure of k, and is often denoted $\overline{k}$; then the condition of being algebraically closed reads $k = \overline{k}$. This is notation I will occasionally slip and use, although we don't really need to dwell on the notion of algebraic closures at the moment.

One last thing to think about: can an algebraically closed field be finite? You are invited to explore this in Exercise 2.2.8. The following lemma might help.

Lemma 1.4.6. Let k be an algebraically closed field. If $f(x) \in k[x]$ is a polynomial such that $f(\alpha) = 0$ for all $\alpha \in k$, then f is the zero polynomial.

Proof. The polynomial $f + 1$ has no roots in k and is hence a constant polynomial. ∎

In fact, the condition of being algebraically closed is sufficient but not necessary; this result is, of course, the one-dimensional analog of Exercise 2.2.6. This result now allows us to prove nonemptiness results for curves.

Theorem 1.4.7. If $C \subset \mathbb{A}^2_k$ is a curve over an algebraically closed field k, then $C(k) \neq \emptyset$.

Proof. Suppose $C = C_f$ for some nonconstant $f(x, y) \in k[x, y]$. Write

$$f(x, y) = a_n(x)y^n + a_{n-1}(x)y^{n-1} + \cdots + a_0(x)$$

for some integer $n \geq 0$ and polynomials $a_0(x), \ldots, a_n(x) \in k[x]$ with $a_n(x) \neq 0$. If $n = 0$, then f is a polynomial of x alone; since f is nonconstant and k is algebraically closed, we may pick a root $\alpha \in k$ of this polynomial and any $\beta \in k$ whatsoever to give us the point $(\alpha, \beta) \in C$. If $n \geq 1$, then Lemma 1.4.6 gives us an $\alpha \in k$ such that $a_n(\alpha) \neq 0$; then the polynomial $f(\alpha, y) \in k[y]$ is nonconstant, so again, since k is alegbraically closed, there is a root $\beta \in k$ of $f(\alpha, y)$, giving us again $(\alpha, \beta) \in C$. ∎

[10] This is a subtlety which we will not have the need to discuss right now, and a true discussion of which belongs to algebra courses anyway.

[11] What would that mean?

This statement–every algebraic curve $C \subset \mathbb{A}_k^2$ is nonempty–is a characterization of algebraically closed fields, although not an awfully useful one. In fact, as you can check, the proof gives us more: the proof above shows that if C is not already the union of finitely many vertical lines, then for all but finitely many values of a (namely the roots of $a_n(x)$, if any), the curve C will intersect the vertical line $x = a$. In particular, if k is infinite (see Exercise 2.2.8), then this argument shows that $C(k)$ must be infinite as well. (So we are leaving behind the nonsense of a curve being finitely many points as well.) In Exercise 2.2.7, you are invited to discuss whether the complement $\mathbb{A}_k^2 \smallsetminus C$ of C in $\mathbb{A}_k^2$ is infinite as well. The picture is therefore somewhat easier to understand over algebraically closed fields than over general fields–this is the reason that we shall essentially restrict ourselves to working with algebraically closed fields from now on.

Example 1.4.8. Considering the hyperbola defined by the vanishing of $f(x, y) = xy - 1$ and taking the line $x = 0$ shows that it is not necessarily true than an algebraic curve C intersects *every* vertical line. Somehow, the point of intersection of $f(x, y) = xy - 1$ with $x = a$ "moves to infinity" as $a \to 0$; this is a situation we will rectify in projective space, where every curve will intersect every other. More on that soon!