1.11 07/03/24 - Projective Duality, (De)Homogenization, Projective Nullstellensatz

Last time, we introduced the projective plane and projective curves. Let's start by looking at an extended example first.

1.11.1 Projective Lines and Projective Duality

Definition 1.11.1. A projective line is a projective curve of the form $L = C_F \subset \mathbb{P}^2_k$ for a nonconstant linear homogeneous polynomial $F \in k[X, Y, Z]_1$.

Once we have the notion of degrees for projective curves below (§1.11.3), then we'll see that a projective line is a projective curve of degree 1.

Example 1.11.2. If $F = Z$, then $L = C_F$ is called the line at infinity and denoted L_∞.

Any linear homogeneous F is specified as $F = AX + BY + CZ$, where $A, B, C \in k$ are not all zero. Note that multiplying F by a nonzero scalar $\lambda \in k^\times$ does not affect C_F. Analogously to the affine case, we will see (in Theorem 1.11.17) that $L = C_F$ recovers F up to nonzero scalars, and hence we get a bijection between the set of lines $L \subset \mathbb{P}^2_k$ and the set of ordered triples (A, B, C) of elements of k, not all zero, subject to the equivalence $(A, B, C) \sim (\lambda A, \lambda B, \lambda C)$ for all $\lambda \in k^*$–but that's just another projective plane! We denote this projective plane by $\mathbb{P}^{2*}_k := \mathbb{P}^2_k(A, B, C)$, so we have a bijection

$$\{\text{lines } L \subset \mathbb{P}^2_k\} \leftrightarrow \mathbb{P}^{2*}_k.$$

Note that points in $\mathbb{P}^{2*}_k$ correspond to lines in $\mathbb{P}^2_k$, but the symmetry of the equation

$$AX + BY + CZ = 0$$

tells us that *lines* in $\mathbb{P}^{2*}_k$ correspond to *points* in $\mathbb{P}^2_k$–and indeed, if a point $P \in \mathbb{P}^2_k$ corresponds to the line $P^* \in \mathbb{P}^{2*}_k$, and the line $L \subset \mathbb{P}^2_k$ corresponds to the point $L^* \in \mathbb{P}^{2*}_k$, then we have

$$P \in L \Leftrightarrow P^* \ni L^*.$$

This funny phenomenon of interchanging the set of lines in one projective plane with the set of points in another is called the phenomenon of projective duality. Duality is a powerful tool that allows us to start with statements about points, lines, and incidences, and produce corresponding "dual" statements–effectively doubling the number of statements we can prove about the projective plane with very little effort. This is because this duality carries with it a lot of structure.

Consider, for instance, the following asymmetry: in $\mathbb{A}^2_k$, given any two points, there is a unique line passing through them, but given any two lines, they either interesect in a unique point or not at all (i.e. if they are parallel). In the projective plane, duality asserts that this asymmetry cannot happen.

Proposition 1.11.3. Given any two distinct points $P_1, P_2 \in \mathbb{P}^2_k$, there is a unique line $L \subset \mathbb{P}^2_k$ through them, and given two distinct lines $L_1, L_2 \subset \mathbb{P}^2_k$, they intersect in a unique point.

Proof. The second assertion follows from the first applied to $\mathbb{P}_k^{2*}$ (i.e. by duality), and so it suffices to show the first one. Suppose we write $P_1 = [X_1 : Y_1 : Z_1]$ and $P_2 = [X_2 : Y_2 : Z_2]$; then we are trying to solve simultaneously the system of equations

$$AX_1 + BY_1 + CZ_1 = 0$$
$$AX_1 + BY_2 + CZ_2 = 0$$

for A, B, C, not all zero, up to scaling. Multiplying the first equation by Y_2 and the second by Y_1 and subtracting yields

$$A(X_1Y_2 - X_2Y_1) + C(Z_1Y_2 - Z_2Y_1) = 0.$$

Similarly, we obtain two other equations of this sort. It follows easily (check!) that there is a solution to the above system of equations, up to scalars, given by

$$[A : B : C] = [Y_1Z_2 - Y_2Z_1 : Z_1X_2 - Z_2X_1 : X_1Y_2 - X_2Y_1],$$

where at least one of the expressions $Y_1Z_2 - Y_2Z_1$, $Z_1X_2 - Z_2X_1$, and $X_1Y_2 - X_2Y_1$ is nonzero because $P_1 \neq P_2$ (why?). ∎

Similarly, the question of collinearity of three points in $\mathbb{P}_k^2$ is answered by

Proposition 1.11.4. Given points $P_1, P_2, P_3 \in \mathbb{P}_k^2$, write $P_i = [X_i : Y_i : Z_i]$ for $i = 1, 2, 3$. The points P_1, P_2 and P_3 are collinear iff

$$\det \begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix} = 0.$$

Proof. The points P_1, P_2, P_3 are collinear iff there are $A, B, C \in k$, not all zero, such that $AX_i + BY_i + CZ_i = 0$ for $i = 1, 2, 3$. This can be rephrased by asking for A, B, C, not all zero, such that

$$\begin{bmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{bmatrix} \begin{bmatrix} A \\ B \\ C \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and then the result follows from simple linear algebra: if the determinant of this matrix were nonzero, it would be invertible (by Cramer's rule, say), and so we would conclude from such an equation that $A = B = C = 0$, and conversely, if the determinant *is* zero, then there is a nonzero vector in the kernel of the linear map determined by it. ∎

Note that projective duality tells us that concurrent triples of lines $L_1, L_2, L_3 \subset \mathbb{P}_k^2$ correspond exactly to collinear triples of points in $\mathbb{P}_k^{2*}$, and we get a corresponding criterion for concurrency of lines, which I will leave to you to formulate.

Of course, this statement automatically implies a corresponding statement in the affine plane (Corollary 1.11.5)) as well, but somehow I have always found the projective ase easier to understand conceptually.

Corollary 1.11.5. Given points $p_1, p_2, p_3 \in \mathbb{A}_k^2$ with coordinates $p_i = (x_i, y_i)$, we have that p_1, p_2, p_3 are collinear iff

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) + (x_1y_2 - x_2y_1) = 0.$$

Proof. The points p_i are collinear in $\mathbb{A}_k^2$ iff the points $P_i := [x_i : y_i : 1]$ are collinear in $\mathbb{P}_k^2$. ■

Remark 1.11.6. Note that similarly to how projective lines in $\mathbb{P}_k^2$ are parametrized by another $\mathbb{P}_k^2$, it is clear that curves $C \subset \mathbb{P}_k^2$ of a fixed degree (interpreted appropriately, i.e. with multiplicity) are also parametrized by a projective space of higher dimension. For instance, a conic section $C \subset \mathbb{P}_k^2$ is specified by a homogeneous quadratic polynomial

$$F = AX^2 + BXY + CY^2 + DXZ + EXZ + FZ^2,$$

which amounts to giving a 6-tuple (A, B, C, D, E, F) of elements of k, not all zero, up to simultaneous scaling: in other words, the set of all conics $C \subset \mathbb{P}_k^2$ is a $\mathbb{P}_k^5$. More generally, the set of all degree $d \geq 1$ curves $C \subset \mathbb{P}_k^2$ is a projective space $\mathbb{P}_k^{d(d+3)/2}$, and even more generally, the set of all degree $d \geq 1$ hypersurfaces $Z \subset \mathbb{P}_k^n$ for $n \geq 1$ is given by a projective space $\mathbb{P}_k^{\binom{d+n}{n}-1}$. (Think about what this could mean–I haven't defined projective spaces of higher dimensions for you yet!) This idea of parameter spaces in our own category is unique to algebraic geometry–for instance, the set of submanifolds of a smooth manifold does not have the structure of a finite-dimensional manifold in any way. This notion of parameter spaces is one of the most powerful tools in modern algebraic geometry: the geometry of a parameter space often dictates the behavior of the objects it parametrizes. I will not dwell on this further, but I would encourage you to think about this as and when this idea shows up in your further studies.

1.11.2 (De)Homogenization, Projective Closure and Affine Part

Let's now start talking about the relationship between affine and projective curves. For this, we first need some algebraic definitions.

Definition 1.11.7.

(a) Given a polynomial $f \in k[x, y]$ of degree $d \geq 0$, we define its homogenization, written f^{h}, to be

$$f^{\mathrm{h}}(X, Y, Z) := Z^d f\left(\frac{X}{Z}, \frac{Y}{Z}\right) \in k[X, Y, Z]_d.$$

In other words, if $f \neq 0$ and we write $f = f_0 + f_1 + \cdots + f_d$ with each f_i homogeneous of degree i, and $f_d \neq 0$, then we have

$$f^{\mathrm{h}}(X, Y, Z) = Z^d f_0(X, Y) + Z^{d-1} f_1(X, Y) + \cdots + f_d(X, Y).$$

(b) Given a homogeneous polynomial $F \in k[X, Y, Z]$, we define the inhomogeneous part or dehomogenization of F with respect to Z, denoted F^{i}, to be

$$F^{\mathrm{i}}(x, y) := F(x, y, 1) \in k[x, y].$$

We will use simple properties of these operations such as $(fg)^{\mathrm{h}} = f^{\mathrm{h}} g^{\mathrm{h}}$ for nonzero $f, g \in k[x, y]$ without further comment. Note that although we have for any $f \in k[x, y]$ that

$(f^{\mathrm{h}})^{\mathrm{i}} = f$, the operations $f \mapsto f^{\mathrm{h}}$ and $F \mapsto F^{\mathrm{i}}$ are not inverse bijections in general. For any nonzero $f \in k[x,y]$ of degree d, the homogenization f^{h} is homogeneous of degree $d = \deg f$, and $Z \nmid f^{\mathrm{h}}$ because $f_d \neq 0$; therefore, if $Z \mid F$, then we cannot possibly have $F = (F^{\mathrm{i}})^{\mathrm{h}}$. However, this is the only problem: we have for any nonzero $F \in k[X,Y,Z]$ that is homogeneous of degree $d \geq 0$ that if $F = Z^m F_0$ for some $m \geq 0$ and $F_0 \in k[X,Y,Z]_{d-m}$ not divisible by d, then $(F^{\mathrm{i}})^{\mathrm{h}} = F_0$, whence $F = Z^m (F^{\mathrm{i}})^{\mathrm{h}}$. In particular, if $Z \nmid F$, then $F = (F^{\mathrm{i}})^{\mathrm{h}}$. Phrased slightly differently, we have

Lemma 1.11.8. For any $d \geq 0$, the operations $f \mapsto f^{\mathrm{h}}$ and $F \mapsto F^{\mathrm{i}}$ give inverse bijections between the set of all nonzero polynomials $f \in k[x,y]$ of degree d and the set of all nonzero homgeneous polynomials $F \in k[X,Y,Z]$ of degree d such that $Z \nmid F$.

The parallel definitions in geometry are as follows.

Definition 1.11.9.

(a) Given an affine curve $C \subset \mathbb{A}^2_k$, we define its projective closure, denoted $\overline{C}$, to be $\overline{C} := C_{f^{\mathrm{h}}}$, where $f \in k[x,y]$ is any polynomial such that $C = C_f$. Given any affine curve $C \subset \mathbb{A}^2_k$, we define the set of points at infinity along C to be $\overline{C} \cap L_\infty$.

(b) Given a projective curve $C \subset \mathbb{P}^2_k$, we define its affine part in the chart defined by $Z \neq 0$ to be $C^\circ := C \cap \mathbb{A}^2_k = \{P \in C : P = [X : Y : Z] \text{ and } Z \neq 0\} = C_{F^{\mathrm{i}}}$, where $F \in k[X,Y,Z]$ is any homogeneous polynomial such that $C = C_F$.

The first thing to note here is that if $f, g \in k[x,y]$ are polynomials such that $C_f = C_g$, then $C_{f^{\mathrm{h}}} = C_{g^{\mathrm{h}}}$, making the projective closure well-defined; similarly, if $F, G \in k[X,Y,Z]$ are homogeneous polynomials such that $C_F = C_G$, then $C_{F^{\mathrm{i}}} = C_{G^{\mathrm{i}}}$ (which is somewhat easier to see from the alternative description). Next, we note that if $C \subset \mathbb{A}^2_k$ has degree $d \geq 1$, then $\overline{C}$ is obtained by attaching at most d new points to C (i.e. there are at most d points at infinity along C); namely, if we write $f = f_0 + \cdots + f_d$, then points of $\overline{C} \setminus C$ correspond to roots of the homogeneous polynomial $f_d(X,Y)$, of which there are at most d distinct values. This observation has the amusing consequence that an algebraic curve of degree d in $\mathbb{A}^2_k$ can have at most d distinct asymptotes.[23] Finally, we have as before that if $C \subset \mathbb{A}^2_k$ is an affine curve, then $(\overline{C})^\circ = C$, but the operations $C \mapsto \overline{C}$ and $C \mapsto C^\circ$ are not inverse bijections: if we consider the line at infinity L_∞, then $L^\circ_\infty = \emptyset$, whence $\overline{L^\circ_\infty} = \emptyset$ as well. Again, this is the only problem, and if $C \subset \mathbb{P}^2_k$ is any projective curve other than L_∞, then C° is a nonempty affine curve. In fact, we have

Lemma 1.11.10. If $C \subset \mathbb{P}^2_k$ is a projective curve, then either $L_\infty \not\subset C$, in which case we have $C = \overline{C^\circ}$, or we have $C = \overline{C^\circ} \cup L_\infty$.

Proof. Left to the reader. ∎

Remark 1.11.11. The terminology "projective closure" comes from topology: there is a topology on $\mathbb{P}^2_k$ called the Zariski topology, in which $\overline{C}$ is just the ordinary topological closure of $C \subset \mathbb{A}^2_k \subset \mathbb{P}^2_k$. Understanding the Zariski topology is absolutely fundamental to appreciating more advanced algebraic geometry, but we don't need to worry too much about it right now.

The goal of this translation is that it allows us to port over the work that we did in the affine case to the projective case without a lot of additional effort. This is what we do now. Let's do a couple of examples.

[23] What are those?

Proposition 1.11.12. If $C, D \subset \mathbb{P}^2_k$ are projective curves, then so is $C \cup D$.

Proof. If $C = C_F$ and $D = C_G$, then $C \cup D = C_{F \cdot G}$. ∎

Proposition 1.11.13. If k is an algebraically closed field and $C \subset \mathbb{P}^2_k$ is a projective curve, then $C = C(k)$ is infinite.

Proof. Either $C = L_\infty$, in which case we are done because k is infinite (how?), or C° is an affine curve, so we are done by Lemma 1.5.1. ∎

Let's now move on to a few more things that follow easily.

1.11.3 Homogeneous Unique Factorization, Nullstellensatz, etc.

Lemma 1.11.14. If $F, G \in k[X, Y, Z]$ are such that F is homogeneous and $G \mid F$, then G is homogeneous.

Proof. Write $F = GH$, and suppose that the degrees of F, G, H are $d, m, n \geq 0$ with $m + n = d$. If $m = 0$ or $n = 0$, then the result is clear; hence assume that $m, n \geq 1$, so $d \geq 2$. Expand $G = G_0 + G_1 + \cdots + G_m$ and $H = H_0 + \cdots + H_n$ with each G_i (resp. each H_j) homogeneous of degree i (resp. j), and $G_m \neq 0$ (resp. $H_n \neq 0$). Let i be the least non-negative integer such that $G_i \neq 0$, so that $0 \leq i \leq m$; similarly, let j be the least non-negative integer such that $H_j \neq 0$. Then the degree $i + j$ component of $F = GH$ is $G_i H_j$, which is nonzero; since we assumed that F is homogeneous of degree d, this implies that $i + j = d$, whence $i = m$ and $j = n$, showing that both G and H are homogeneous. ∎

From this, we immediately obtain a homogeneous analog of unique factorization in $k[X, Y, Z]$, namely

Theorem 1.11.15 (Homogeneous Unique Factorization)**.** Every nonconstant homogeneous $F \in k[X, Y, Z]$ can be factored as

$$F = F_1 \cdots F_n,$$

a product of finitely many homogeneous irreducible elements $F_1, \ldots, F_n \in k[X, Y, Z]$, and this factorization is unique up to the order of the elements and multiplication by units.

I will leave to you to make the last statement precise (say along the lines of Definition 1.5.7.)

Proof. Immediate consequence of unique factorization in $k[X, Y, Z]$ (Corollary 1.5.14) and the Lemma 1.11.14 above. ∎

Now we can mimic the affine theory as follows. Firstly, the analog of Theorem 1.6.6 is

Theorem 1.11.16 (Projective Finite Intersection)**.** Let $F, G \in k[X, Y, Z]$ be nonconstant relatively prime homogeneous polynomials. Then $C_F \cap C_G$ is finite.

Proof. Note that Z cannot divide both F and G; without loss of generality, suppose that $Z \nmid G$. Since

$$C_F \cap C_G \subset (C_F^\circ \cap C_G^\circ) \cup (L_\infty \cap C_G),$$

it suffices to show that both $C_F^\circ \cap C_G^\circ$ and $L_\infty \cap C_G$ are finite. The latter is easy: if $\deg G = d \geq 0$, and we write

$$G = Z^d G_0(X,Y) + Z^{d-1} G_1(X,Y) + \cdots + G_d(X,Y),$$

where each $G_j(X,Y) \in k[X,Y]_j$ is homogeneous of degree d, then $Z \nmid G$ implies that $G_d \neq 0$, whence $L_\infty \cap C_G$ corresponds to the finitely many roots of the homogeneous polynomial $G_d(X,Y)$, of which there are at most d.[24] To show the former, note that $C_F^\circ \cap C_G^\circ = C_{F^{\mathrm{i}}} \cap C_{G^{\mathrm{i}}}$, so in light of Theorem 1.6.6, it suffices to show that if $F, G \in k[X,Y,Z]$ are nonconstant relatively prime homogeneous polynomials, then the dehomogenizations $F^{\mathrm{i}}, G^{\mathrm{i}} \in k[x,y]$ are also relatively prime (although no longer necessarily nonconstant). To show this statement, it suffices note that if $q \in k[x,y]$ is such that $q | F^{\mathrm{i}}$, then $F^{\mathrm{i}} = pq$ for some $p \in k[x,y]$, whence $q^{\mathrm{h}} \mid p^{\mathrm{h}} q^{\mathrm{h}} = (F^{\mathrm{i}})^{\mathrm{h}} \mid F$; then, if a nonconstant $q \in k[x,y]$ were to divide both F^{i} and G^{i}, then the nonconstant[25] $q^{\mathrm{h}} \in k[X,Y,Z]$ would divide F and G, contradicting their relative primality. ■

This theorem was the key to the Nullstellensatz, and all of its corollaries, which we collect in one theorem here.

Theorem 1.11.17 (Projective Nullstellensatz). Suppose that k is an algebraically closed field.

(a) If $F, G \in k[X,Y,Z]$ are nonconstant homogeneous polynomials, then $C_G \subset C_F$ iff there is some integer $n \geq 1$ such that $G \mid F^n$.
(b) If $F, G \in k[X,Y,Z]$ are nonconstant homogeneous polynomials with F irreducible, then $C_G \subset C_F$ implies $C_G = C_F$.
(c) If $F \in k[X,Y,Z]$ is a nonconstant homogeneous polynomial, then C_F is irreducible.[a] Conversely, if $C \subset \mathbb{P}_k^2$ is an irreducible projective curve, then there is an irreducible homogeneous $F \in k[X,Y,Z]$ such that $C = C_F$.

[a]You were invited to define the notion of irreducibility for projective curves in Exercise 2.4.2.

Proof.

(a) Identical to the proof of Theorem 1.6.7: if Q is a prime factor of G, then Q is homogeneous by Lemma 1.11.14, and then if Q and F were relatively prime, then $C_Q \cap C_F = C_Q$ would be finite by Theorem 1.11.16 but infinite by Proposition 1.11.13.
(b) Identical to the proof of Corollary 1.6.8 using (a) instead of Theorem 1.6.7.
(c) Identical to the proof of Theorem 1.5.6, and left to the reader.

■

Similarly to the affine case, given a projective curve $C \subset \mathbb{P}_k^2$, we can try to define a vanishing ideal $\mathbb{I}(C) \subset k[X,Y,Z]$ of C consisting of homogeneous polynomials vanishing on C, but the problem is that the sum of two homogeneous polynomials of different degrees is not homogeneous. The correct definition is

[24]In other words, we have $[X_0 : Y_0 : Z_0] \in L_\infty \cap C_G$ iff $Z_0 = 0$ and $G_d(X_0, Y_0) = 0$, but the latter condition constrains the ratio $[X_0 : Y_0]$ to be one of the homogeneous roots of $G_d(X_0, Y_0)$, i.e. if we factor G_d using Lemma 1.8.3 (and Theorem 1.4.5 if needed) as $G_d = \prod_{i=1}^d (\lambda_i X + \mu_i Y)$, then $[X_0 : Y_0]$ can only be one of the d possible choices for $[-\mu_i : \lambda_i]$.
[25]This uses $\deg q^{\mathrm{h}} = \deg q$.

Definition 1.11.18. Given a projective curve $C \subset \mathbb{P}^2_k$, we define the vanishing ideal of C to be

$$\mathbb{I}(C) := \{F \in k[X,Y,Z] : \text{if } F = F_0 + \cdots + F_d \text{ with } F_j \in k[X,Y,Z]_j \text{ then } C \subset C_{F_j} \text{ for all } j.\}$$

This is, in fact, an ideal of $k[X,Y,Z]$–and, indeed, a special kind of ideal called a homogeneous ideal.[26] Then the analog of Theorem 1.6.12 still holds: $\mathbb{I}(C)$ is a principal ideal generated by $\mathrm{rad}(F)$ for any homogeneous $F \in k[X,Y,Z]$ such that $C = C_F$. A generator of $\mathbb{I}(C)$ is again called a minimal polynomial of C; any two of these differ by a nonzero scalar, and we define the degree of C to be the degree of any minimal polynomial for C. The analog of Corollary 1.6.13 still holds: over $k = \overline{k}$, there is a bijective correspondence between projective curves $C \subset \mathbb{P}^2_k$ and principal ideals of $k[X,Y,Z]$ generated by nonconstant reduced homogeneous $F \in k[X,Y,Z]$, and the curve C is irreducible iff $\mathbb{I}(C)$ is a prime ideal. Finally, we also have an analog of Theorem 1.7.10; let's write this down in some detail.

Theorem 1.11.19 (Projective Unique Decomposition)**.** If $k = \overline{k}$, then given any curve $C \subset \mathbb{P}^2_k$, there is an integer $n \geq 1$ and irreducible curves $C_1, \ldots, C_n \subset \mathbb{P}^2_k$ such that $C_i \neq C_j$ for $i \neq j$, such that

$$C = C_1 \cup C_2 \cup \cdots \cup C_n.$$

The integer n is uniquely determined, as are the C_j up to reordering.

Proof. Identical to the proof of Theorem 1.7.10. ■

The curves $C_1, \ldots, C_n \subset C$ occuring in such a decomposition are called the irreducible comopnents of C. Finally, the analog of Theorem 1.7.11 is

Theorem 1.11.20 (Projective Finite Intersection Revisited)**.** If $C, D \subset \mathbb{P}^2_k$ are two curves that don't share any common irreducible components, then the intersection $C \cap D$ is finite.

Proof. Identical to the proof of Theorem 1.7.11. ■

The three things from the affine case that we haven't transferred yet are (a) parametric curves, (b) changes of coordinates, and (c) (intersection) multiplicity. This we will do in the next two lectures.

1.11.4 Addendum: Irreducible Projective Curves

I did not have time to cover this in lecture, but I do want to explain the relationship between minimal polynomials and irreducibility of affine curves and their projective counterparts. This is the content of

[26] Can you come up with a good definition of this notion?

Theorem 1.11.21.

(a) If $f \in k[x, y]$ is irreducible (resp. reduced), then so is $f^{\mathrm{h}} \in k[X, Y, Z]$. Conversely, if a homogeneous $F \in k[X, Y, Z]$ is irreducible (resp. reduced), then so is $F^{\mathrm{i}} \in k[x, y]$, unless $F = \lambda Z^m$ for some $\lambda \in k^\times$ and $m \geq 0$ (resp. $m = 0, 1$), in which case, and only in which case, $F^{\mathrm{i}} = \lambda$ is a nonzero constant.

(b) If an affine curve $C \subset \mathbb{A}^2_k$ has minimal polynomial f, then its projective closure $\overline{C}$ has minimal polynomial f^{h}; in particular, $\deg C = \deg \overline{C}$. If $C \subset \mathbb{P}^2_k$ has minimal polynomial F, then its affine part C°, if nonempty, has minimal polynomial F^{i} and either

   (i) $L_\infty \subset C$ and $\deg C^\circ = \deg C - 1$ (where $\deg C^\circ = 0$ says just that $C^\circ = \emptyset$), or

   (ii) $L_\infty \not\subset C$ and $\deg C^\circ = \deg C$.

(c) If $C \subset \mathbb{A}^2_k$ is an irreducible affine curve, then its projective closure $\overline{C}$ is an irreducible projective curve. If $C \subset \mathbb{P}^2_k$ is an irreducible projective curve, then either $C^\circ = \emptyset$ (which happens iff $C = L_\infty$), or C° is an irreducible affine curve.

Proof.

(a) Let's treat irreducibility; the proof for reducedness is similar and left to the reader. If given an $f \in k[x, y]$, there is a $G \in k[X, Y, Z]$ such that $G \mid f^{\mathrm{h}}$ and $0 < \deg G < \deg f^{\mathrm{h}} = \deg f$, then G is homogeneous by Lemma 1.11.14 and $Z \nmid G$ because $Z \nmid f^{\mathrm{h}}$, from which we get that $G^{\mathrm{i}} \mid (f^{\mathrm{h}})^{\mathrm{i}} = f$ and $0 < \deg G^{\mathrm{i}} = \deg G < \deg f$; therefore, if f is irreducible, then so is f^{h}. Conversely, given a homogeneous $F \in k[X, Y, Z]$ that is not of the form λZ^m, we must have $\deg F^{\mathrm{i}} \geq 1$; if $g \in k[x, y]$ is such that $g \mid F^{\mathrm{i}}$ and $0 < \deg g < \deg F^{\mathrm{i}} \leq \deg F$, then $g^{\mathrm{h}} \mid (F^{\mathrm{i}})^{\mathrm{h}} \mid F$ with $0 < \deg g^{\mathrm{h}} = \deg g < \deg F$; therefore, if F is irreducible, then so is F^{i}.

(b) If an affine curve C has minimal polynomial f, then f is reduced, and so by (a) so is f^{h}; since f^{h} is a reduced homogeneous polynomial vanishing on $\overline{C}$, it follows that f^{h} is a minimal polynomial for $\overline{C}$. I will leave the rest to the reader.

(c) If C is an irreducible affine curve, then any minimal polynomial f for C is irreducible; then f^{h} is irreducible by (a) and a minimal polynomial for $\overline{C}$ by (b), and so it follows that $\overline{C}$ is an irreducible projective curve. The converse is again left to the reader.

■

Finally, the symmetry in X, Y, Z tells us that irreducibility of a given homogeneous $F \in k[X, Y, Z]$ is testable by dehomogenization with respect to any of the variables.